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Continuity properties for modulation spaces, with applications to pseudo-differential calculus—I

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Abstract

Let $M^{p,q}$ denote the modulation space with parameters $p,q \in [1,\infty]$. If $1/p_1 + 1/p_2 = 1 + 1/p_0$ and $1/q_1 + 1/q_2 = 1/q_0$, then it is proved that $M^{p_1,q_1} * M^{p_2,q_2} \subset M^{p_0,q_0}$. The result is used to get inclusions between modulation spaces, Besov spaces and Schatten classes in calculus of Ψdo (pseudo-differential operators), and to extend the definition of Toeplitz operators. We also discuss continuity of ambiguity functions and Ψdo in the framework of modulation spaces.

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0. Introduction

During the period 1980–1983, Feichtinger introduced in [8,10] the modulation spaces, denoted by $M^{p,q}$, $p,q \in [1,\infty]$, as an appropriate family of function and distribution spaces to have in background when discussing certain problems within time–frequency analysis. In this topic, the ambiguity function, which is connected to the Wigner distribution and the short time Fourier transform, is an important concept. The modulation spaces, obtained by imposing a mixed $L^{p,q}$-norm on

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ambiguity functions, give opportunities to consider certain decay and propagation properties at infinity for distributions, as well as certain localization properties. In [10,12–14,18], Feichtinger and Gröchenig developed the basic theory for modulation spaces. Important parts in these investigations concern invariance and continuity properties, for example embeddings and convolutions for modulation spaces. One did also analyze the ambiguity function using modulation space theory.

The theory for such spaces has thereafter been developed in different ways. In [17], Gröbner obtained embeddings between modulation spaces and Besov spaces, and in [26], Okoudjou found embeddings between modulation spaces and Sobolev/Triebel spaces. We also note that some other convolution results may be found in [19].

During the last 10 years, after a systematic development of the modulation space theory already had been done by Feichtinger and Gröchenig, the modulation spaces have also occurred in the theory of pseudo-differential operators, and supply the topic with symbol classes which are defined without any explicit reference to derivatives. In the early paper [29] by Sjöstrand, algebraic and continuity questions for pseudo-differential operators with symbols in $M^{\infty,1}$ are treated. Here, for example, it is proved that such operators are L^2-continuous. In [3], Boulkhemair extends this result to a large class of Fourier integral operators. Independently by Boulkhemair [3] and Sjöstrand [29], Gröchenig and Heil prove in [19,20], that such pseudo-differential operators are continuous on any modulation space $M^{p,q}$, which in particular also implies L^2-continuity, since $M^{2,2} = L^2$. These results also immediately apply to pseudo-differential operators with symbols in S_0^0, since $S_0^0 \subset M^{\infty,1}$. Here, and in what follows, S_0^0 denotes the Hörmander class which consists of all smooth functions which are bounded together with all their derivatives. (In many situations, the notation $S_{0,0}^0$ is used instead of S_0^0.) We finally remark that related results might be found in [6,23,25,31,34].

The aim of this paper is to study certain continuity questions for modulation spaces, and to apply the results to the theory of pseudo-differential operators and Toeplitz operators. The continuity investigations consist mainly of finding convolution properties between modulation spaces and Lebesgue spaces, and to find embeddings between modulation spaces and Besov spaces. Some attention is also paid to continuity questions for ambiguity functions in the context of modulation spaces. The applications to pseudo-differential calculus are of different kinds. From the embeddings between modulation spaces and Besov spaces we obtain embeddings between Schatten–von Neumann classes in pseudo-differential calculus and Besov spaces. Our investigations of the ambiguity function may also, in a way similar to [19,20], be applied to pseudo-differential calculus using modulation spaces. This means that the symbols for the pseudo-differential operators should belong to modulation spaces, and that continuity for such operators is discussed in background of modulation spaces. Recall that the symbol classes in classical theory, in contrast to the modulation spaces, are usually formulated in terms of restrictions on derivatives, and continuity is usually discussed in the context of Sobolev spaces, and not in the context of modulation spaces. (Cf. [22].)

The applications to the theory of Toeplitz operators are essentially made within the Weyl calculus of pseudo-differential operators, using the fact that the Weyl symbol of a Toeplitz operator may be expressed as a convolution of the Toeplitz symbol and a Wigner distribution. Then it is straightforward to apply the convolution results for modulation spaces, mentioned above.

We remark also that the paper is based on the research report [36].

Let us now briefly discuss our results. We start by recalling the definition of modulation spaces. Assume that $p, q \in [1, \infty]$, and that $\chi \in \mathscr{S}(\mathbf{R}^m)\backslash 0$, and set $\tau_x \chi = \chi(\cdot - x)$. (We use the same notations for the usual function and distribution spaces, e.g. in [22].) The modulation space $M^{p,q}(\mathbf{R}^m)$ consists of all $a \in \mathscr{S}'(\mathbf{R}^m)$ such that

$$\left(\int \left(\int |\mathscr{F}(a\tau_x\chi)(\xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q} < \infty, \tag{0.1}$$

(with obvious modification when $p = \infty$ or $q = \infty$). Here $\mathscr{F}$ denotes a choice of Fourier transform. (In [30], the set $M^{\infty,1}$ is denoted by S_w, and in [34], the notation S_w^p is used instead of $M^{p,1}$.) The space $M^{p,q}$ increases with the parameters p and q. The smallest space, $M^{1,1}$ is contained in L^p and $\mathscr{F}L^p$ for every p, which in turn are contained in $M^{\infty,\infty}$. As for Sobolev spaces, weighted modulation spaces are obtained by including multiplicative weight-functions in the integrand of (0.1). (Cf. Chapter 11 in [19].)

In Section 2, Young-type results for modulation spaces and Lebesgue spaces are discussed. More precisely we prove that the usual convolution product on $\mathscr{S}$ extends to a continuous mapping from $M^{p_1,q_1} \times M^{p_2,q_2}$ to M^{p_0,q_0}, provided that $1/p_1 + 1/p_2 = 1 + 1/p_0$ and $1/q_1 + 1/q_2 = 1/q_0$. If in addition $q_j = 1$ when $j = 0$ or $q_j = \infty$ when $j = 1$ or $j = 2$, then the extension also holds when M^{p_j,q_j} above is replaced by L^{p_j}. This means that $M^{p_1,q_1} * M^{p_2,q_2} \subset L^{p_0}$ when $q_0 = 1$ and $M^{p_1,q_1} * L^{p_2} \subset M^{p_0,q_0}$ when $q_2 = \infty$. The basic ideas for these results and their verifications are rather similar with Theorem 3 and its proof in [9], which implies that $M^{p_1,q_1} \cdot M^{p_2,q_2} \subset M^{p_0,q_0}$, when $1/p_1 + 1/p_2 = 1/p_0$ and $1/q_1 + 1/q_2 = 1 + 1/q_0$.

In Section 3 we apply the convolution results above, in order to prove embedding relations between modulation spaces and Besov spaces. Here some ideas from Section 2 in [35] are used, where embeddings between Schatten–von Neumann classes in Weyl calculus and Besov spaces are proved. Since there are certain narrow embedding properties between Sobolev spaces and Besov spaces, we obtain at the same time inclusions between Sobolev spaces and modulation spaces. The results of these investigations are summarized in Theorem 3.1 below, from which it follows that if

$$\begin{aligned} &\theta_1(p,q) = \max(0, q^{-1} - \min(p^{-1}, p'^{-1})), \\ &\theta_2(p,q) = \min(0, q^{-1} - \max(p^{-1}, p'^{-1})), \quad p, q \in [1, \infty], \end{aligned} \tag{0.2}$$

then

$$H^p_{\mu m \theta_1(p,q)}(\mathbf{R}^m) \subset M^{p,q}(\mathbf{R}^m) \subset H^p_{\mu m \theta_2(p,q)}(\mathbf{R}^m), \quad \text{when } \mu > 1. \tag{0.3}$$

Here H_s^p is the Sobolev space of distributions with s derivatives in L^p, and p' denotes the conjugate exponent to p, i.e. p and p' satisfy $1/p + 1/p' = 1$. In particular, some of the results in [26] are obtained. For example, if $2 \leqslant p \leqslant \infty$ and $q = p'$, then the first inclusion in (0.3) is equivalent to Corollary 1 in Section 3 in [26]. Theorem 3.1 also improves Theorem F.4 in [17] by P. Gröbner, which deals with embeddings between α-modulation spaces and Besov spaces.

These inclusions also give embedding results for pseudo-differential operators with low regularity assumptions on their symbols, which we shall describe now. Assume that $a \in \mathscr{S}(\mathbf{R}^{2m})$ and that $t \in \mathbf{R}$. Then the pseudo-differential operator $a_t(x, D)$, with symbol a, on $\mathscr{S}(\mathbf{R}^m)$ is defined by

$$
\begin{aligned}
(a_t(x,D)f)(x) &= (\mathrm{Op}_t(a)f)(x) \\
&= (2\pi)^{-m} \int\int a((1-t)x + ty, \xi) f(y) e^{i\langle x-y, \xi\rangle}\, dy\, d\xi,
\end{aligned}
$$

where $f \in \mathscr{S}(\mathbf{R}^m)$. The definition extends to any $a \in \mathscr{S}'(\mathbf{R}^{2m})$, in which case $a_t(x, D)$ becomes a continuous operator from $\mathscr{S}(\mathbf{R}^m)$ to $\mathscr{S}'(\mathbf{R}^m)$. (Cf. [22] or [34].) If $t = 1/2$, then the Weyl quantization is obtained, and we write $a^w(x, D)$ and $\mathrm{Op}^w(a)$ instead of $a_{1/2}(x, D)$ and $\mathrm{Op}_{1/2}(a)$ respectively.

We let $s_{t,p}(\mathbf{R}^{2m})$ denote the set of all $a \in \mathscr{S}'(\mathbf{R}^{2m})$ such that $a_t(x, D)$ is a Schatten–von Neumann operator on $L^2(\mathbf{R}^m)$ of order $p \in [1, \infty]$. The inclusion relations above then also give inclusions between $s_{t,p}$-spaces, Besov spaces and Sobolev spaces. We prove for example that for every $\mu > 1$ and $p \in [1, \infty]$, then

$$
H^p_{2\mu m|1-2/p|}(\mathbf{R}^{2m}) \subseteq s_{t,p}(\mathbf{R}^{2m}) \subseteq H^p_{-2\mu m|1-2/p|}(\mathbf{R}^{2m}).
$$

(Note that if $t = 0$ or $t = 1/2$, then for certain p, sharper inclusions are presented in [2,4,35].)

In the last section we make some further applications of the convolution results in Section 2 to the theory of pseudo-differential operators and Toeplitz operators. These investigations are partially based on certain continuity properties for the ambiguity function

$$
W_{f,g}(x, \xi) = (2\pi)^{-m/2} \int f(y/2 - x) g(y/2 + x) e^{i\langle y, \xi\rangle}\, dy, \tag{0.4}
$$

and its relation to Weyl operators. Here $f, g \in \mathscr{S}(\mathbf{R}^m)$, which implies that $W_{f,g} \in \mathscr{S}(\mathbf{R}^{2m})$. The definition extends to any $f, g \in \mathscr{S}'(\mathbf{R}^m)$, and then $W_{f,g} \in \mathscr{S}'(\mathbf{R}^{2m})$ (cf. [16,32] or [35]).

We present sufficient conditions on f and g, in order to $W_{f,g}$ should belong to certain modulation spaces, and prove for example that if $p \leqslant q$, $f \in M^{p,q}(\mathbf{R}^m)$ and $g \in M^{q,p}(\mathbf{R}^m)$, then $W_{f,g} \in M^{p,q}(\mathbf{R}^{2m})$. As an application, we improve Theorem 14.5.2 in [19] and give, for certain p and q, sufficient conditions on $p_j, q_j, j = 1, 2$, such that $a_t(x, D)$ is continuous from M^{p_1,q_1} to M^{p_2,q_2}, when $a \in M^{p,q}$ and $t \in \mathbf{R}$. As a consequence of these investigations, we also obtain continuity results of a more classical character, and discuss S_0^0-continuity and L^p-continuity for $a_t(x,D)$ when $a \in S_0^0(\mathbf{R}^{2m})$.

In order to explain our results concerning Toeplitz operators we recall the definition of these operators. For any fixed $h_0 \in \mathscr{S}(\mathbf{R}^m)\backslash 0$, and any $a \in \mathscr{S}'(\mathbf{R}^{2m})$, the Toeplitz operator $\mathrm{Tp}_{h_0}(a)$ is defined by the formula

$$(\mathrm{Tp}_{h_0}(a)f, g) = (a(-2\cdot)W_{f,h_0}, W_{g,h_0}) \tag{0.5}$$

for every $f, g \in \mathscr{S}(\mathbf{R}^m)$. Here $(\cdot,\cdot)$ is the usual L^2-scalar product. In Section 4 we also consider the case when h_0 belongs to certain modulation spaces.

The theory of Toeplitz operators may be considered as a part of the Weyl calculus, since the Weyl symbol for $\mathrm{Tp}_{h_0}(a)$ is equal to $a * u_{h_0}$, where $u_{h_0} = (2\pi)^{-m/2} W_{\tilde{h}_0,h_0}$. Here, and in what follows, we set $\tilde{f}(x) = \overline{f(-x)}$. (Cf. [27,32, Section 5.2] or [35, Section 4].) We may then apply the convolution results in Section 2 to $a * u_{h_0}$, and it turns out that $\mathrm{Tp}_{h_0}(a)$ is a Schatten–von Neumann operator of order p when $a \in M^{p,q}(\mathbf{R}^{2m})$ for some $q \in [1,\infty]$.

1. Preliminaries

In this section we present some basic properties of modulation spaces and their relations to Lebesgue spaces and Schatten–von Neumann spaces in the pseudo-differential calculus. Here the most of the results may be found in for example [15,17] or [19]. We start by making a review of certain invariance and growth properties of the modulation spaces.

Assume that $p, q \in [1,\infty]$ and that $\chi \in \mathscr{S}(\mathbf{R}^m)\backslash 0$. Then we recall that $M^{p,q}(\mathbf{R}^m)$ is the set of all $a \in \mathscr{S}'(\mathbf{R}^m)$ such that $H_{a,p,\chi} \in L^q(\mathbf{R}^m)$, where

$$H_{a,p}(\xi) = H_{a,p,\chi}(\xi) \equiv \left(\int |\mathscr{F}(a\tau_x\chi)(\xi)|^p \, dx \right)^{1/p}. \tag{1.1}$$

Here $\mathscr{F}$ is some choice of Fourier transform. We also set

$$||a||_{M^{p,q}} = ||a||_{M^{p,q,\chi}} \equiv ||H_{a,p,\chi}||_{L^q}. \tag{1.2}$$

If $p = q$, then the notations $M^p(\mathbf{R}^m)$ and $||\cdot||_{M^p}$ are used instead of $M^{p,p}(\mathbf{R}^m)$ and $||\cdot||_{M^{p,p}}$, respectively.

The set $M^{p,q}(\mathbf{R}^m)$ is a Banach space which is independent of the choice of $\mathscr{F}$ and χ above, and different choice of χ give rise to equivalent norms (cf. [15, Section 3.2.2] or [19, Chapter 11]).

Remark 1.1. In [34], the independent properties of certain types of modulation spaces are further investigated. More precisely, assume that μ is a non-negative periodic Borel measure, and that $\chi\in\mathscr{S}(\mathbf{R}^m)$ satisfies

$$\int \chi(y-x)\,d\mu(y)\neq 0 \quad \text{for every } x\in\mathbf{R}^m. \tag{1.3}$$

Assume also $p\in[1,\infty]$ and $q=1$. Then Theorem 2.7 in [34] asserts that $a\in M^{p,q}(\mathbf{R}^m)$, if and only if

$$H_{a,p,\chi,d\mu}(\xi)\equiv\left(\int |\mathscr{F}(a\tau_x\chi)(\xi)|^p\,d\mu(x)\right)^{1/p}$$

belongs to $L^q(\mathbf{R}^m)$. By similar arguments it follows that the same conclusion holds for every, $p,\ q\in[1,\infty]$. Moreover, $||a||=||H_{a,p,\chi,d\mu}||_{L^q}$ defines a norm equivalent to any $M^{p,q}$-norm at the above.

Note that if $d\mu$ is the Lebesgue measure, then the conclusion in Remark 1.1 holds for any $\chi\in\mathscr{S}\backslash 0$. For general periodic, non-negative and non-trivial Borel measure μ, it seems to be an open question for the author whether condition (1.3) may be replaced by the weaker condition $\chi\in\mathscr{S}\backslash 0$ such that the conclusion in Remark 1.1 still holds.

In the most situations it is assumed that $d\mu(x)=dx$, while the choice of χ depends on the situation under consideration. We also note that there are no problems with measureability in (1.1), since for any $a\in\mathscr{S}'(\mathbf{R}^m)$ and any $\chi\in\mathscr{S}(\mathbf{R}^m)$, then $(x,\xi)\mapsto\mathscr{F}(a\tau_x\chi)(\xi)$ is a smooth function.

Obviously, $M^{p,q}(\mathbf{R}^m)$ is independent of the choice of Fourier transform $\mathscr{F}$, and from now on it is assumed that $\mathscr{F}$ is given by

$$\mathscr{F}f(\xi)=\hat{f}(\xi)=\pi^{-m/2}\int f(x)e^{-2i\langle x,\xi\rangle}\,dx. \tag{1.4}$$

It follows from this definition that for admissible $f,g\in\mathscr{S}'(\mathbf{R}^m)$, we have

$$\mathscr{F}(f*g)=\pi^{m/2}\hat{f}\hat{g} \quad\text{and}\quad \mathscr{F}(fg)=\pi^{-m/2}\hat{f}*\hat{g}. \tag{1.5}$$

We shall next consider the space $M^{p,q}_{\mathscr{F}}(\mathbf{R}^m)\equiv\mathscr{F}M^{p,q}(\mathbf{R}^m)$, when $p,q\in[1,\infty]$. Such spaces are examples on Wiener amalgam spaces, and we recall that $M^{p,q}$ and $\mathscr{F}M^{p,q}$ agree, if and only if $p=q$. (Cf. [8,19].) On the other hand, the definition of the $M^{p,q}_{\mathscr{F}}$-spaces may be expressed in a similar way as the definition of the modulation spaces. Indeed, Parseval's formula gives

$$|\mathscr{F}(\hat{a}\,\tau_\xi\hat{\chi})(x)|=|\mathscr{F}(a\tau_{-x}\check{\chi})(\xi)|. \tag{1.6}$$

Here and in what follows, $\check{\chi}(x)=\chi(-x)\in\mathscr{S}(\mathbf{R}^m)\backslash 0$. It follows now from (1.6) that $M^{p,q}_{\mathscr{F}}(\mathbf{R}^m)$ consists of all $a\in\mathscr{S}'(\mathbf{R}^m)$ such that

$$||a||_{M^{p,q,\chi}_{\mathscr{F}}}\equiv\left(\int\left(\int |\mathscr{F}(a\tau_x\chi)(\xi)|^p\,d\xi\right)^{q/p}dx\right)^{1/q}$$

is finite. Moreover, different choice of such χ give rise to equivalent norms, and $\|a\|_{M^{p,q,\hat{\chi}}_{\mathscr{F}}} = \|\mathscr{F} a\|_{M^{p,q,\chi}}$.

We shall next introduce a few notations. Assume that B_1 and B_2 are topological vector spaces. Then we use the notation $B_1 \hookrightarrow B_2$ when B_1 is continuously embedded in B_2. This means, when B_1 and B_2 are Banach spaces, that $B_1 \subset B_2$, and that $\|x\|_{B_2} \leqslant C\|x\|_{B_1}$, for some constant $C>0$ independent on $x \in B_1$.

We let $\langle \cdot, \cdot \rangle$ denote the canonical dual form between functions or distributions spaces and their duals, and we set $(a,b) = \langle a, \bar{b} \rangle$, for admissible a and b in $\mathscr{S}'(\mathbf{R}^m)$. It is then clear that $(\cdot,\cdot)$ on L^2 is the usual scalar product.

We recall that a quadratic form Φ on $\mathbf{R}^m$ is called non-degenerate when the determinant of its corresponding symmetric matrix is non-zero. In view of Section 2 in [34] it is clear that if Φ is a real-valued and non-degenerate, then the map $a \mapsto e^{i\Phi} * a$ extends to a homeomorphism on $\mathscr{S}'$. The following proposition is an immediate consequence of Lemma 1.2(iii) in [8], Subsection 3.2.2 in [15], Section 11.3 in [19], and Proposition 2.14 and its proof in [34]. Some motivations may also be found in [36]. Here we recall that if $p \in [1,\infty]$, then its conjugate exponent, $p' \in [1,\infty]$, satisfies $1/p + 1/p' = 1$.

Proposition 1.2. *Assume that $p, q \in [1,\infty]$. Then the following are true*:

(1) *if $p_1, p_2, q_1, q_2 \in [1,\infty]$ such that $p_1 \leqslant p_2$ and $q_1 \leqslant q_2$, then $\mathscr{S}(\mathbf{R}^m) \hookrightarrow M^{p_1,q_1}(\mathbf{R}^m) \hookrightarrow M^{p_2,q_2}(\mathbf{R}^m) \hookrightarrow \mathscr{S}'(\mathbf{R}^m)$;*
(2) *the map $(a,b) \mapsto \langle a,b \rangle$ from $\mathscr{S}(\mathbf{R}^m) \times \mathscr{S}(\mathbf{R}^m)$ to $\mathbf{C}$ extends to a continuous bilinear map from $M^{p,q}(\mathbf{R}^m) \times M^{p',q'}(\mathbf{R}^m)$ to $\mathbf{C}$. On the other hand, if $\|a\| = \sup |\langle a,b \rangle|$, where the supremum is taken over all $b \in M^{p',q'}(\mathbf{R}^m)$ such that $\|b\|_{M^{p',q'}} \leqslant 1$, then $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_{M^{p,q}}$;*
(3) *if $p, q < \infty$, then $\mathscr{S}(\mathbf{R}^m)$ is dense in $M^{p,q}(\mathbf{R}^m)$, and the dual space of $M^{p,q}(\mathbf{R}^m)$ may be identified with $M^{p',q'}(\mathbf{R}^m)$, through the form $\langle \cdot,\cdot \rangle$. Moreover, $\mathscr{S}(\mathbf{R}^m)$ is weakly dense in $M^{\infty}(\mathbf{R}^m)$. Here and in* (2), *a similar result holds when $\langle \cdot,\cdot \rangle$ is replaced by $(\cdot,\cdot)$;*
(4) *the Fourier transform is a homeomorphism on $M^p(\mathbf{R}^m)$;*
(5) *if Φ is a real-valued non-degenerate quadratic form on $\mathbf{R}^m$, then the map $a \mapsto T_\Phi a = e^{i\Phi} * a$ is a homeomorphism on $M^{p,q}(\mathbf{R}^m)$. Moreover, if $d\mu(x) = dx$ and $\chi = T_\Phi \psi$, where $\psi \in \mathscr{S}(\mathbf{R}^m)$, then $\chi \in \mathscr{S}(\mathbf{R}^m)$ and $\|a\|_{M^{p,q,\chi}} = \|T_\Phi a\|_{M^{p,q,\psi}}$.*

Continuity discussions for modulation spaces are usually dependent of the possibilities to approximate elements in $M^{p,q}$ with elements in C_0^∞. A primary idea which might appear is to use the usual norm convergence. This causes however unconvenient obstacles when $p = \infty$ or $q = \infty$, since C_0^∞ is not dense in $M^{p,q}$ for such p and q. In the case $1 \leqslant p \leqslant \infty$ and $q < \infty$, this annoying detail may be avoided by using a slight generalization of the narrow convergence, presented in [29,30,34].

Definition 1.3. Assume that $p, q \in [1, \infty]$ and that $a, a_j \in M^{p,q}(\mathbf{R}^m)$, $j = 1, 2, \dots$. Then a_j converges *narrowly* to a (with respect to p, q, χ and $d\mu$) as j turns to infinity, if the following conditions are fulfilled:

(1) $a_j \to a$ in $\mathscr{S}'(\mathbf{R}^m)$ as $j \to \infty$;
(2) $H_{a_j,p,\chi,d\mu}(\xi) \to H_{a,p,\chi,d\mu}(\xi)$ in $L^q(\mathbf{R}^m)$ as $j \to \infty$;

The following result follows by a straightforward modification of the proof of Proposition 2.3 in [34].

Proposition 1.4. *Assume that $d\mu_1, \dots, d\mu_N$ are non-negative periodic Borel measures on $\mathbf{R}^m$, and that the functions $\chi_1, \dots, \chi_N \in \mathscr{S}(\mathbf{R}^m)$ satisfy $\int \chi_j(y - x)\, d\mu_j(y) \neq 0$ for every $x \in \mathbf{R}^m$ and every j. Assume also that $p, q \in [1, \infty]$ such that $q < \infty$. Then for every $a \in M^{p,q}(\mathbf{R}^m)$, there is a sequence $\{a_k\}$ in $C_0^\infty(\mathbf{R}^m)$ which converges narrowly to a with respect to p, q, χ_j and $d\mu_j$, when $1 \leqslant j \leqslant N$. In particular, $C_0^\infty(\mathbf{R}^m)$ is narrowly dense in $M^{p,q}(\mathbf{R}^m)$.*

From now on we assume that $d\mu(x) = dx$. We shall next recall some facts in operator theory and certain symbol classes in pseudo-differential calculus. Let $\mathscr{I}_p$ be the Banach space of Schatten–von Neumann operators on $L^2(\mathbf{R}^m)$ of order $p \in [1, \infty]$, i.e. $\mathscr{I}_p$ consists of all operators T on $L^2(\mathbf{R}^m)$ such that

$$||T||_{\mathscr{I}_p} \equiv \sup\Big(\sum |(Tf_j, g_j)|^p\Big)^{1/p}$$

is finite. Here the supremum should be taken over all orthonormal sequences (f_j) and (g_j) in $L^2(\mathbf{R}^m)$. Note that $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_\infty$ are the sets of trace-class, Hilbert–Schmidt and continuous operators, respectively. We refer to [28] for more facts about the $\mathscr{I}_p$-spaces.

Recall from the introduction that if $t \in \mathbf{R}$ then $s_{t,p}(\mathbf{R}^{2m})$ is the set of all $a \in \mathscr{S}'(\mathbf{R}^{2m})$ such that $a_t(x, D) \in \mathscr{I}_p$, and set $||a||_{s_{t,p}} = ||a_t(x, D)||_{\mathscr{I}_p}$. The standard representation is obtained when $t = 0$ and then the notation $a(x, D)$ and $\mathrm{Op}(a)$ are used instead of $a_0(x, D)$ and $\mathrm{Op}_0(a)$ respectively. Recall also that in the case $t = 1/2$, the Weyl quantization $a^w(x, D) = \mathrm{Op}^w(a)$ is obtained, and in this case the notation $s_p^w(\mathbf{R}^{2m})$ is used instead of $s_{t,p}(\mathbf{R}^{2m})$.

If $t \in \mathbf{R}$ is fixed, then the map $a \mapsto a_t(x, D)$, from $\mathscr{S}'(\mathbf{R}^{2m})$ to the set of all continuous linear operators from $\mathscr{S}(\mathbf{R}^m)$ to $\mathscr{S}'(\mathbf{R}^m)$, is a bijection. (Cf. [22] or [34].) This implies that the map $a \mapsto a_t(x, D)$ from $s_{t,p}(\mathbf{R}^{2m})$ to $\mathscr{I}_p$ is an isometric homeomorphism. In particular, $s_{t,p}$ is a Banach space for every $t \in \mathbf{R}$ and $p \in [1, \infty]$.

Remark 1.5. Note that the pseudo-differential calculi $a_1 \mapsto \mathrm{Op}_{t_1}(a_1)$ and $a_2 \mapsto \mathrm{Op}_{t_2}(a_2)$ may be carried over to each others in a canonical way, for every choice of $t_1, t_2 \in \mathbf{R}$. In fact, if $a_1, a_2 \in \mathscr{S}'(\mathbf{R}^{2m})$, then

$$\mathrm{Op}_{t_1}(a_1) = \mathrm{Op}_{t_2}(a_2) \;\Leftrightarrow\; a_2(x, \xi) = (e^{i(t_2 - t_1)\langle D_x, D_\xi \rangle} a_1)(x, \xi).$$

(Cf. [34] or [22].) In the case $t_1 \neq t_2$, then $a_2 = e^{i\Phi} * a_1$, for some non-degenerate and real quadratic form Φ on $\mathbf{R}^{2m}$. Hence Proposition 1.2(5) implies that if $p, q \in [1, \infty]$ and $t \in \mathbf{R}$, then the following are true:

(1) $\mathrm{Op}_{t_1}(M^{p,q}(\mathbf{R}^{2m})) = \mathrm{Op}_{t_2}(M^{p,q}(\mathbf{R}^{2m}))$, and the map $e^{it\langle D_x, D_\xi\rangle}$ is a homeomorphism on $M^{p,q}(\mathbf{R}^{2m})$;
(2) the map $a \mapsto e^{i(t_2-t_1)\langle D_x, D_\xi\rangle} a$ is an isometric homeomorphism from $s_{t_1,p}(\mathbf{R}^{2m})$ to $s_{t_2,p}(\mathbf{R}^{2m})$.

Since $\mathrm{Op}_t(M^{p,q})$ is independent of $t \in \mathbf{R}$ by Remark 1.4, the notation $\mathrm{Op}(M^{p,q})$ will be used instead of $\mathrm{Op}_t(M^{p,q})$. By similar reasons, the notation $\mathrm{Op}(S_0^0)$ is used instead of $\mathrm{Op}_t(S_0^0)$. (Cf. [7, Chapter 7].) In the following lemma we list some basic properties for the $s_{t,p}$-spaces.

Lemma 1.6. *Assume that $t \in \mathbf{R}$. Then the following are true*:

(1) *if $p_1, p_2 \in [1, \infty]$ such that $p_1 \leqslant p_2$, then*

$$\mathscr{S}(\mathbf{R}^m) \hookrightarrow s_{t,p_1}(\mathbf{R}^{2m}) \hookrightarrow s_{t,p_2}(\mathbf{R}^{2m}) \hookrightarrow \mathscr{S}'(\mathbf{R}^m);$$

(2) *the scalar product $(\cdot,\cdot)$ on $\mathscr{S}(\mathbf{R}^{2m})$ extends to a continuous form on $s_{t,p}(\mathbf{R}^{2m}) \times s_{t,p'}(\mathbf{R}^{2m})$. For every $a \in s_{t,p}(\mathbf{R}^{2m})$ and $b \in s_{t,p'}(\mathbf{R}^{2m})$, it follows that*

$$|(a,b)| \leqslant (2\pi)^m \|a\|_{s_{t,p}} \|b\|_{s_{t,p'}}, \quad \|a\|_{s_{t,p}} (2\pi)^{-m} \sup |(a,c)|,$$

where the supremum should be taken over all $c \in s_{t,p'}(\mathbf{R}^{2m})$ such that $\|c\|_{s_{t,p'}} \leqslant 1$.

If in addition $p < \infty$, then the dual space for $s_{t,p}(\mathbf{R}^{2m})$ may be identified with $s_{t,p'}(\mathbf{R}^{2m})$ through the form $(\cdot,\cdot)$;
(3) *$\mathscr{S}(\mathbf{R}^{2m})$ is dense in $s_{t,p}(\mathbf{R}^{2m})$ for every $p < \infty$, and dense in $s_{t,\infty}(\mathbf{R}^{2m})$ with respect to the weak* topology*;
(4) *$s_{t,2}(\mathbf{R}^{2m}) = L^2(\mathbf{R}^{2m})$ and $\|a\|_{s_{t,2}} = (2\pi)^{-m/2} \|a\|_{L^2}$ for every $a \in L^2(\mathbf{R}^{2m})$.*

Proof. In the case $t = 1/2$, the lemma follows from Section 1 in [33] or [35]. For general t, the result is now a consequence of Remark 1.5 and that the map $a \mapsto e^{it\langle D_x, D_\xi\rangle} a$ is a self-adjoint unitary operator on $L^2(\mathbf{R}^{2m})$, which is homeomorphic on $\mathscr{S}(\mathbf{R}^{2m})$ and on $\mathscr{S}'(\mathbf{R}^{2m})$. The proof is complete. □

In view of Lemma 1.6(4) we also note that for $p \in \{1, \infty\}$, there is no simple characterization for the s_p^w-spaces in terms of Lebesgue spaces or Sobolev/Besov spaces. We refer to [32,33] or [35] for more facts about these spaces. In the following result, which in many parts is a consequence of [17], Corollary 3.5 in [20] and

Corollary 3.2.7 in [15], we make comparisons between modulation spaces, Lebesgue-spaces and $s_{t,p}$-spaces.

Proposition 1.7. *Assume that* $p, q, r \in [1, \infty]$, *and that* $t \in \mathbf{R}$. *Then the following are true*:

(1) *if* $q \leqslant p \leqslant q'$ *and* $p \leqslant r$, *then* $M^{p,q}(\mathbf{R}^{2m}) \hookrightarrow s_{t,r}(\mathbf{R}^{2m})$;
(2) *if* $q' \leqslant p \leqslant q$ *and* $r \leqslant p$, *then* $s_{t,r}(\mathbf{R}^{2m}) \hookrightarrow M^{p,q}(\mathbf{R}^{2m})$;
(3) *if* $q \leqslant p \leqslant r \leqslant q'$, *then* $M^{p,q}(\mathbf{R}^m) \hookrightarrow L^r(\mathbf{R}^m)$;
(4) *if* $q' \leqslant r \leqslant p \leqslant q$, *then* $L^r(\mathbf{R}^m) \hookrightarrow M^{p,q}(\mathbf{R}^m)$;
(5) *if* $p \leqslant q \leqslant r \leqslant p'$, *then* $M^{p,q}(\mathbf{R}^m) \hookrightarrow \mathscr{F}L^r(\mathbf{R}^m)$;
(6) *if* $p' \leqslant r \leqslant q \leqslant p$, *then* $\mathscr{F}L^r(\mathbf{R}^m) \hookrightarrow M^{p,q}(\mathbf{R}^m)$.

Moreover, *if the condition on r in any of assertions* (3)–(6) *above are replaced by the opposite condition, then the corresponding inclusion fails to hold.*

In Section 3 we present inclusions between Besov spaces and modulation spaces, in which slight refinements comparing to Proposition 1.7 are obtained.

Proof. (1) By Theorem 1.5 and Proposition 1.6 in [34], the result is true for $1 \leqslant p \leqslant \infty$ and $q = 1$. In the case $p = q = 2$, the assertion is obviously true since $s_{t,2} = L^2 = M^2$, with equivalent norms. The result follows now for general p and q by interpolation. (Cf. Section 2 in [35] and Corollary 2.3 in [8].)

Assertion (2) follows from (1) and duality. (Cf. Proposition 1.1 and Lemma 1.6.)

Next we prove (5). We note that Corollary 3.2.7 in [15] together with the fact that M^1 is invariant under Fourier transformation, implies that $M^1 \hookrightarrow L^1 \cap \mathscr{F}L^1$. If $p = q = 2$, then $r = 2$ and again the assertion follows in this case from the fact that $M^2 = L^2 = \mathscr{F}L^2$.

Assume next that $p = 1$ and $q = \infty$, and choose $\chi \in \mathscr{S}(\mathbf{R}^m)$ such that $\int \chi(y)\, dy = 1$. Then $r = \infty$, and we get

$$|\hat{a}(\xi)| = \left| \int \mathscr{F}(a\tau_x\chi)(\xi)\, dx \right| \leqslant \int |\mathscr{F}(a\tau_x\chi)(\xi)|\, dx.$$

If we take the supremum of the left-hand side, then we obtain $\|\hat{a}\|_{L^\infty} \leqslant \|a\|_{M^{1,\infty,\chi}}$, which proves that $M^{1,\infty} \hookrightarrow \mathscr{F}L^\infty$. The result follows now from these estimates and interpolation (cf. Corollary 2.3 in [8]).

By (5) and duality, we get (6). Assertion (3) and (4) now follows from (5), (6) and Theorem 3.2 in [8], which asserts that $\mathscr{F}M^{q,p} \subset M^{p,q}$ when $p \leqslant q$. The last part of the proposition is obtained by choosing λ in appropriate ways in the following lemma. $\square$

Lemma 1.8. *Assume that* $u_\lambda(x) = e^{-\lambda|x|^2}$ *and that* $\chi(x) = e^{-|x|^2}$ *when* $x \in \mathbf{R}^m$. *Then*

$$\|u_\lambda\|_{M^{p,q,\chi}} = \pi^{m(1/p+1/q)/2} p^{-m/2p} q^{-m/2q} \lambda^{-m/2p} (1+\lambda)^{m(1/p+1/q-1)/2}.$$

Proof. The assertion follows by some straightforward computations. $\square$

There are, of course, other interesting inclusion properties which are not stated in Proposition 1.7. It was, for example, remarked already in [29] that $M^{\infty,1}(\mathbf{R}^m)$ contains $S_0^0(\mathbf{R}^m)$.

We shall end this section by showing that there is a natural way to extend the form $\langle \cdot, \cdot \rangle$ on $\mathscr{S} \times \mathscr{S}$ to a continuous bilinear form on $M^{\infty,1} \times M^{1,\infty}$, or $M^{1,\infty} \times M^{\infty,1}$.

We note that $\mathscr{S}$ is neither dense in $M^{\infty,1}$ nor in $M^{1,\infty}$, which implies that there are problems with uniqueness when extending $\langle \cdot, \cdot \rangle$. If $a, b, \chi \in \mathscr{S}$ such that $||\chi||_{L^2} = 1$, then $\langle a, b \rangle = \int a(x)b(x)\,dx$, and it follows from Parseval's formula that

$$\langle a, b \rangle = \int\int \mathscr{F}(a\tau_x\chi)(\xi)\mathscr{F}(b\tau_x\bar{\chi})(-\xi)\,dx\,d\xi. \tag{1.7}$$

In the case $a \in M^{\infty,1}$ and $b \in M^{1,\infty}$ we note that the right-hand side makes sense since the L^1-norm of $(x,\xi) \mapsto \mathscr{F}(a\tau_x\chi)(\xi)\mathscr{F}(b\tau_x\bar{\chi})(-\xi)$ is bounded by $||a||_{M^{\infty,1,\chi}}||b||_{M^{1,\infty,\bar{\chi}}}$. For $a \in M^{\infty,1}$ and $b \in M^{1,\infty}$ we therefore define the requested bilinear form on $M^{\infty,1} \times M^{1,\infty}$ by (1.7). By similar arguments, an extension of $\langle \cdot, \cdot \rangle$ to $M^{1,\infty} \times M^{\infty,1}$ is obtained. The following lemma shows that these definitions are quite natural.

Lemma 1.9. *Assume that $a \in M^{\infty,1}(\mathbf{R}^m)$ and $b \in M^{1,\infty}(\mathbf{R}^m)$. Then the following are true*:

(1) $|\langle a, b \rangle| \leqslant ||a||_{M^{\infty,1,\chi}}||b||_{M^{1,\infty,\bar{\chi}}}$;
(2) *if $a \in \mathscr{S}$ or $b \in \mathscr{S}$, then $\langle a, b \rangle$ is the usual dual form between a distribution and a test function*;
(3) *if $a_j \in M^{\infty,1}$, $j = 1, 2, \ldots$, converges narrowly to a as $j \to \infty$, then $\langle a_j, b \rangle \to \langle a, b \rangle$ as $j \to \infty$*;
(4) *the form $\langle \cdot, \cdot \rangle$ on $M^{\infty,1}(\mathbf{R}^m) \times M^{1,\infty}(\mathbf{R}^m)$ does not depend on $\chi \in \mathscr{S}$ such that $||\chi||_{L^2} = 1$*;
(5) $\langle a, b \rangle = \langle b, a \rangle$.

Proof. Conditions (1) and (2) follows immediately from (1.7), Hölder's inequality and Parseval's formula.

(3) Let $a_0 = a$. Then it follows from the narrow convergence that $\mathscr{F}(a_j\tau_x\chi)(\xi) \to \mathscr{F}(a_0\tau_x\chi)(\xi)$ pointwise for every $(x,\xi) \in \mathbf{R}^{2m}$ as $j \to \infty$. Moreover, if $U(x,\xi) = |\mathscr{F}(b\tau_x\bar{\chi})(-\xi)|$, then

$$|\mathscr{F}(a_j\,\tau_x\chi)(\xi)\mathscr{F}(b\tau_x\chi)(-\xi)| \leqslant H_{a_j,\infty,\chi}(\xi)U(x,\xi) \in L^1(\mathbf{R}^{2m}).$$

Since $a_j \to a$ narrowly as $j \to \infty$, it follows that

$$||(H_{a_j,\infty,\chi} - H_{a,\infty,\chi})U||_{L^1} \leqslant ||H_{a_j,\infty,\chi} - H_{a,\infty,\chi}||_{L^1}||b||_{M^{1,\infty,\bar{\chi}}} \to 0,$$

as $j \to \infty$. It follows now that

$$\mathscr{F}(a_j\tau_x\chi)(\xi)\mathscr{F}(b\tau_x\chi)(-\xi) \to \mathscr{F}(a\tau_x\chi)(\xi)\mathscr{F}(b\tau_x\chi)(-\xi)$$

in $L^1(\mathbf{R}^{2m})$ as $j \to \infty$, by a generalization of Lebesgue's theorem which asserts that if $f_j \to f$ a.e. as $j \to \infty$ and if there exists a sequence $g_j \in L^1$ such that $|f_j| \leqslant g_j$ and $\|g_j - g_k\|_{L^1} \to 0$ as $j, k \to \infty$, then $\|f - f_j\|_{L^1} \to 0$ as $j \to 0$. This implies that $\langle a_j, b\rangle \to \langle a, b\rangle$ as $j \to \infty$, and (3) follows.

(4) Assume that $\psi \in \mathscr{S}(\mathbf{R}^m)$ such that $\|\psi\|_{L^2} = 1$. Then, by Proposition 1.4, there is a sequence $a_j \in C_0^\infty(\mathbf{R}^m)$ which converges narrowly to a with respect to χ and ψ, as $j \to \infty$. The result follows now from (2) and (3). From the narrow convergence we also obtain (5). The proof is complete. $\square$

Remark 1.10. It follows from Proposition 1.4 and Lemma 1.9 that the extension of the form $\langle \cdot, \cdot \rangle$ from $\mathscr{S} \times \mathscr{S}$ to $M^{\infty,1} \times M^{1,\infty}$ is unique if, for any fixed $b \in M^{1,\infty}$, we require that the map $a \mapsto \langle a, b\rangle$ from $M^{\infty,1}$ to $\mathbf{C}$ should be continuous with respect to the narrow convergence.

2. Young type inequalities for modulation spaces and Lebesgue spaces

In this section we extend the usual convolution on $\mathscr{S}$ to a multiplication between appropriate modulation spaces and Lebesgue spaces. The requirements on the involving spaces is that their coefficients should satisfy certain Hölder and Young conditions. More precisely, we prove that if $p_j, q_j \in [1, \infty]$ where $j = 0, 1, 2$ such that $1/p_1 + 1/p_2 = 1 + 1/p_0$ and $1/q_1 + 1/q_2 = 1/q_0$, then $M^{p_1,q_1} * M^{p_2,q_2} \subset M^{p_0,q_0}$. We also combine this result with Proposition 1.7 in order to obtain convolution results, where modulation spaces, Lebesgue spaces and $s_{t,p}$-spaces are involved.

In most of the cases, the extensions are unique, depending on the fact that $\mathscr{S}$ is dense in $M^{p,q}$ when $p, q < \infty$ and weakly dense in M^∞. The case $p_j = \infty$ and/or $q_j = \infty$, for more than one choice of $j \neq 0$ in the convolution, causes however problems with uniqueness, because $\mathscr{S}$ is then not dense in at least two factors in the convolution.

We start by proving that the usual multiplication on $\mathscr{S}(\mathbf{R}^m)$ extends to a multiplication from $M_{\mathscr{F}}^{p_1,q_1}(\mathbf{R}^m) \times \cdots \times M_{\mathscr{F}}^{p_N,q_N}(\mathbf{R}^m)$ to $M_{\mathscr{F}}^{p_0,q_0}(\mathbf{R}^m)$, provided $p_j \in [1, \infty]$ and $q_j \in [1, \infty]$, for every $0 \leqslant j \leqslant N$, satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_N} = N - 1 + \frac{1}{p_0}, \quad \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_N} = \frac{1}{q_0}. \tag{2.1}$$

Assume also that

$$\chi_0, \chi_1, \ldots, \chi_N \in \mathscr{S}(\mathbf{R}^m) \quad \text{such that} \quad \int \left(\prod_{j=0}^{N} \chi_j(x) \right) dx = 1. \tag{2.2}$$

For every $a_1,\dots,a_N,\varphi\in\mathscr{S}(\mathbf{R}^m)$, it follows from Parseval's formula and (1.5) that

$$\begin{aligned}&\langle a_1\cdots a_N,\varphi\rangle=\pi^{-(N-1)m/2}\int\int(f_{1,x}*\cdots*f_{N,x})(\xi)g_x(-\xi)\,dx\,d\xi,\\ &\text{where}\quad f_{j,x}(\xi)=\mathscr{F}(a_j\tau_x\chi_j)(\xi)\ \text{and}\ g_x(\xi)=\mathscr{F}(\varphi\tau_x\chi_0)(\xi).\end{aligned}\tag{2.3}$$

Here and in what follows, the convolution of $f_{j,x}(\xi)$ should be taken only with respect to the ξ-variable. The following lemma is the first step for using (2.3) in order to extend the convolution product to appropriate modulation spaces.

Lemma 2.1. *Assume that $p_j,q_j\in[1,\infty]$, $0\leqslant j\leqslant N$, satisfy* (2.1), *and that $a_j\in M^{p_j,q_j}_{\mathscr{F}}(\mathbf{R}^m)$ when $1\leqslant j\leqslant N$. Let also $f_{j,x}$ be as in* (2.3). *Then $G(x,\xi)=(|f_{1,x}|*\cdots*|f_{N,x}|)(\xi)$ is measurable and*

$$\left(\int\left(\int(|f_{1,x}|*\cdots*|f_{N,x}|(\xi))^{p_0}\,d\xi\right)^{q_0/p_0}dx\right)^{1/q_0}\leqslant\prod_{j=1}^N||a_j||_{M^{p_j,q_j,\chi_j}_{\mathscr{F}}}.$$

Proof. Since $(x,\xi)\mapsto f_{j,x}(\xi)$ is continuous, the result follows if we first apply Young's inequality with respect to the ξ-variables and then Hölder's inequality with respect to the x-variables on $G(x,\xi)$. $\square$

If $p_j,q_j\in[1,\infty]$, $j=0,\dots,N$, satisfy (2.1), and $a_j\in M^{p_j,q_j}_{\mathscr{F}}(\mathbf{R}^m)$, when $1\leqslant j\leqslant N$, then we take (2.3) as the definition of $a_0=a_1\cdots a_N$ as an element in $\mathscr{S}'(\mathbf{R}^m)$. It follows from Lemma 2.1 that this definition makes sense. From the same lemma, it might not be far away to suspect that indeed $a_0\in M^{p_0,q_0}_{\mathscr{F}}$, which we shall prove next. In order to establish this, we first need to discuss some invariance properties, and prove that

$$\mathscr{F}(a_1\cdots a_N\tau_x(\chi_1\cdots\chi_N))(\xi)=\pi^{-(N-1)m/2}(f_{1,x}*\cdots*f_{N,x})(\xi),\tag{2.4}$$

with equality in $\mathscr{S}'(\mathbf{R}^{2m})$. Here $a_1\cdots a_N$ on the left-hand side is the distribution a_0 above, while the right-hand side is a well defined and measurable function in view of Lemma 2.1. In particular, the left-hand is equal to $\langle a_0,e^{-2i\langle\cdot,\xi\rangle}\tau_x(\chi_1\cdots\chi_N)\rangle$.

We start to prove that the definition of a_0 is independent of the choice of $\chi_0,\dots,\chi_N$ in (2.2).

Lemma 2.2. *Assume that a_j, $1\leqslant j\leqslant N$, are the same as in Lemma* 2.1. *Then $a_1\cdots a_N$ is independent of the choice of $\chi_0,\chi_1,\dots,\chi_N$ in* (2.2).

Proof. We prove the result in the case $N=2$ and $\chi_0=\chi_1=\chi_2=\chi$, leaving the general case to the reader. Assume that $\psi\in\mathscr{S}(\mathbf{R}^m)$ satisfies $\int\psi(x)^3\,dx=1$, and set $\chi_x=\chi(\cdot-x)$ and $\psi_y=\psi(\cdot-y)$. Then the map $(x,y,\xi)\mapsto\mathscr{F}(\varphi\chi_x\psi_y)(\xi)$ belongs to

$\mathscr{S}(\mathbf{R}^{3m})$, which implies that

$$I = \int\int\int (\mathscr{F}(a_1\chi_x\psi_y) * \mathscr{F}(a_2\chi_x\psi_y))(\xi)\mathscr{F}(\varphi\chi_x\psi_y)(-\xi)\, dx\, dy\, d\xi$$
$$= \int\int\int\int \mathscr{F}(a_1\chi_x\psi_y)(\xi)\mathscr{F}(a_2\chi_x\psi_y)(\eta)\mathscr{F}(\varphi\chi_x\psi_y)(-\xi-\eta)\, dx\, dy\, d\xi\, d\eta$$

makes sense. Here we have used Lemma 2.1 and its proof in order to conclude that the integrand on the right-hand side belongs to $L^1(\mathbf{R}^{4m})$. This also implies that we may change the orders of integration. Hence Parseval's formula applied to the ξ-variables gives

$$I = \int\int\int\int (a_1\chi_x\psi_y)(x_1)\mathscr{F}(a_2\chi_x\psi_y)(\eta)(\varphi\chi_x\psi_y)(x_1)e^{2i\langle x_1,\eta\rangle}\, dx\, dy\, dx_1\, d\eta$$
$$= \int\int\int\int (a_1\chi_x)(x_1)\mathscr{F}(a_2\chi_x\psi_y)(\eta)(\varphi\chi_x\psi_y^2)(x_1)e^{2i\langle x_1,\eta\rangle}\, dx\, dy\, dx_1\, d\eta$$
$$= \int\int\int\int \mathscr{F}(a_1\chi_x)(\xi)\mathscr{F}(a_2\chi_x\psi_y)(\eta)\mathscr{F}(\varphi\chi_x\psi_y^2)(-\xi-\eta)\, dx\, dy\, d\xi\, d\eta.$$

By similar arguments we get

$$I = \int\int\int\int \mathscr{F}(a_1\chi_x)(\xi)\mathscr{F}(a_2\chi_x)(\eta)\mathscr{F}(\varphi\chi_x\psi_y^3)(-\xi-\eta)\, dx\, dy\, d\xi\, d\eta$$
$$= \int\int\int \mathscr{F}(a_1\chi_x)(\xi)\mathscr{F}(a_2\chi_x)(\eta)\mathscr{F}(\varphi\chi_x)(-\xi-\eta)\, dx\, d\xi\, d\eta,$$

where in the last step we have used that $\int \psi_y^3(x)\, dy = 1$. This gives

$$I = \int\int (\mathscr{F}(a_1\chi_x) * \mathscr{F}(a_2\chi_x))(\xi)\mathscr{F}(\varphi\chi_x)(-\xi)\, dx\, d\xi,$$

which is equal to $\pi^{m/2}\langle a_1a_2, \varphi\rangle$. In the same way we get

$$I = \int\int (\mathscr{F}(a_1\psi_y) * \mathscr{F}(a_2\psi_y))(\xi)\mathscr{F}(\varphi\psi_y)(-\xi)\, dy\, d\xi.$$

This proves the announced invariance, and the proof is complete. $\square$

Lemma 2.3. *Assume that a_j, χ_0, χ_j and $f_{j,x}$, $1 \leqslant j \leqslant N$, are the same as in Lemma* 2.1. *Then* (2.4) *holds with equality in $\mathscr{S}'(\mathbf{R}^{2m})$.*

Proof. We restrict ourselves to the case $N = 2$ and $\chi_0 = \chi_1 = \chi_2 = \chi$. The general case follows by similar arguments and is again left for the reader. We also let $\chi_x = \chi(\cdot - x)$, and we set $F_0(x,\xi) = \pi^{m/2}\mathscr{F}(a_1a_2\chi_x^2)(\xi)$. If $\Phi \in \mathscr{S}(\mathbf{R}^{2m})$, then $\langle F_0, \Phi\rangle = \pi^{m/2}\langle a_1a_2, \varphi\rangle$, where $\varphi(y) = \int \chi(y-x)^2\Phi_2(x,y)\, dx$. Here Φ_2 is the partial Fourier transform of $\Phi(x,\xi)$ with respect to the ξ-variable. This gives

$$\begin{aligned}\langle F_0,\Phi\rangle &= \int\int\int \mathscr{F}(a_1\chi_y)(\xi-\eta)\mathscr{F}(a_2\chi_y)(\eta)\mathscr{F}(\varphi\chi_y)(-\xi)\,dy\,d\xi\,d\eta\\ &= \int\int\int\int \mathscr{F}(a_1\chi_y)(\xi-\eta)\mathscr{F}(a_2\chi_y)(\eta)\mathscr{F}(\chi_y\chi_x^2\Phi_2(x,\cdot))(-\xi)\,dx\,dy\,d\xi\,d\eta.\end{aligned}$$

Here we note again that the integrals make sense since their integrands belong to L^1. Hence using the same arguments as in the proof of Lemma 2.2 for moving the factors χ_x and χ_y, we get that the last integral is equal to

$$\begin{aligned}&\int\int\int\int \mathscr{F}(a_1\chi_x)(\xi-\eta)\mathscr{F}(a_2\chi_x)(\eta)\mathscr{F}(\chi_y^3\Phi_2(x,\cdot))(-\xi)\,dx\,dy\,d\xi\,d\eta\\ &\quad= \int\int\int \mathscr{F}(a_1\chi_x)(\xi-\eta)\mathscr{F}(a_2\chi_x)(\eta)\Phi(x,\xi)\,dx\,d\xi\,d\eta\\ &\quad= \int\int (\mathscr{F}(a_1\chi_x)*\mathscr{F}(a_2\chi_x))(\xi)\Phi(x,\xi)\,dx\,d\xi.\end{aligned}$$

In the first equality we have used that

$$\int \mathscr{F}(\chi_y^3\Phi_2(x,\cdot))(-\xi)\,dy = \mathscr{F}(\Phi_2(x,\cdot))(-\xi) = \Phi(x,\xi),$$

by Fourier's inversion formula, and that $\int \chi(x)^3\,dx = 1$. This gives (2.4), and the proof is complete. $\square$

We note that from (2.2) and Lemma 2.2 it follows that $a_1\cdots a_N$ does not depend on the order of the a_j, when a_j are as above. We are now ready to establish the following result.

Theorem 2.4. *Assume that* $p_j,q_j\in[1,\infty]$ *when* $0\leqslant j\leqslant N$ *satisfy* (2.1). *Then* $(a_1,\dots,a_N)\mapsto a_1\cdots a_N$ *is a continuous, symmetric and associative map from* $M_{\mathscr{F}}^{p_1,q_1}(\mathbf{R}^m)\times\cdots\times M_{\mathscr{F}}^{p_N,q_N}(\mathbf{R}^m)$ *to* $M_{\mathscr{F}}^{p_0,q_0}(\mathbf{R}^m)$. *If* $\chi_0,\chi_j\in\mathscr{S}(\mathbf{R}^m)$ *and* $a_j\in M_{\mathscr{F}}^{p_j,q_j}(\mathbf{R}^m)$ *when* $1\leqslant j\leqslant N$ *such that* $\chi_0=\chi_1\cdots\chi_N$, *then*

$$||a_1\cdots a_N||_{M_{\mathscr{F}}^{p_0,q_0,\chi_0}}\leqslant \pi^{-(N-1)m/2}\prod_{j=1}^N ||a_j||_{M_{\mathscr{F}}^{p_j,q_j,\chi_j}}. \tag{2.5}$$

Here and in other situations, the map $(a_1,\dots,a_N)\mapsto a_1\cdots a_N$ is called associative if the corresponding operation $\cdot$ is associative.

Proof. Estimate (2.5) is an immediate consequence of Lemmas 2.1 and 2.3, which proves the continuity assertion. We also note that the stated symmetry properties follows from (2.3) and Lemma 2.1.

It remains to prove the asserted associativity, which follows if we prove that $a_1 \cdots a_N = (a_1 \cdots a_{j_0})(a_{j_0+1} \cdots a_N)$ holds for every choice of $1 \leqslant j_0 \leqslant N$, when $a_j \in M_{\mathscr{F}}^{p_j,q_j}$ for every j. We shall again restrict ourselves and only prove that $a_1 a_2 a_3 = (a_1 a_2) a_3$, leaving the general case for the reader.

Let $\chi \in \mathscr{S}$ such that $\int \chi(x)^4 \, dx = 1$, and set $\chi_x = \chi(\cdot - x)$ as before. Then (2.3) and Lemma 2.2 gives that

$$\begin{aligned} &\langle a_1 a_2 a_3, \varphi \rangle \\ &\quad = \pi^{-m} \int \int (\mathscr{F}(a_1 \chi_x) * \mathscr{F}(a_2 \chi_x) * \mathscr{F}(a_3 \chi_x))(\xi) \mathscr{F}(\varphi \chi_x)(-\xi) \, dx \, d\xi. \end{aligned}$$

In the same way we get that

$$\langle (a_1 a_2) a_3, \varphi \rangle = \pi^{-m/2} \int \int (\mathscr{F}(a_1 a_2 \chi_x^2) * \mathscr{F}(a_3 \chi_x))(\xi) \mathscr{F}(\varphi \chi_x)(-\xi) \, dx \, d\xi.$$

It follows now from Lemma 2.3 and its proof that we may replace $\mathscr{F}(a_1 a_2 \chi_x^2)$ in the last integral by $\pi^{-m/2} \mathscr{F}(a_1 \chi_x) * \mathscr{F}(a_2 \chi_x)$. This gives the desired equality and completes the proof. $\square$

We shall next discuss extensions of the usual convolution product on $\mathscr{S}$ to certain modulation spaces. We note that if $a_j \in \mathscr{S}(\mathbf{R}^m)$, for every $1 \leqslant j \leqslant N$, then

$$(a_1 * \cdots * a_N)(\xi) = \pi^{(N-1)m/2} \mathscr{F}(\hat{a}_1 \cdots \hat{a}_N)(-\xi). \tag{1.5$'$}$$

In the case $a_j \in M^{p_j,q_j}(\mathbf{R}^m)$, where $p_j, q_j \in [1, \infty]$ when $1 \leqslant j \leqslant N$ satisfies (2.1) for some $p_0, q_0 \in [1, \infty]$, we take (1.5$'$) as the definition of $a_1 * \cdots * a_N$. Here $\hat{a}_1 \cdots \hat{a}_N$ should be interpreted as a product between elements in $M_{\mathscr{F}}^{p_j,q_j}$, which was defined earlier. From Theorem 2.4 it follows that $a_1 * \cdots * a_N$ belongs to $\mathscr{F}(M_{\mathscr{F}}^{p_0,q_0}) = M^{p_0,q_0}$. The following result is therefore an immediate consequence of Theorem 2.4.

Theorem 2.5. *Assume that $p_j, q_j \in [1, \infty]$ when $0 \leqslant j \leqslant N$ satisfy* (2.1). *Then $(a_1, \dots, a_N) \mapsto a_1 * \cdots * a_N$ is a continuous, symmetric and associative map from $M^{p_1,q_1}(\mathbf{R}^m) \times \cdots \times M^{p_N,q_N}(\mathbf{R}^m)$ to $M^{p_0,q_0}(\mathbf{R}^m)$. If $\chi_j \in \mathscr{S}(\mathbf{R}^m)$ and $a_j \in M^{p_j,q_j}(\mathbf{R}^m)$ when $0 \leqslant j \leqslant N$ such that $\chi_0 = \chi_1 * \cdots * \chi_N$, then*

$$||a_1 * \cdots * a_N||_{M^{p_0,q_0,\chi_0}} \leqslant \pi^{(N-1)m/2} \prod_{j=1}^{N} ||a_j||_{M^{p_j,q_j,\chi_j}}. \tag{2.6}$$

From Proposition 1.7 and Theorem 2.5, we also obtain the following.

Theorem 2.6. *Assume that $N_0 \geqslant 0$, and that $p_j, q_j \in [1, \infty]$ when $0 \leqslant j \leqslant N$ satisfy* (2.1), *and that $q_j \geqslant \max(p_j, p_j')$ when $N_0 + 1 \leqslant j \leqslant N$. Then $(a_1, \dots, a_N) \mapsto a_1 * \cdots * a_N$ is a* *continuous, symmetric and associative map from*

$$M^{p_1,q_1}(\mathbf{R}^m)\times\cdots\times M^{p_{N_0},q_{N_0}}(\mathbf{R}^m)\times L^{p_{N_0+1}}(\mathbf{R}^m)\times\cdots\times L^{p_N}(\mathbf{R}^m) \tag{2.7}$$

to $M^{p_0,q_0}(\mathbf{R}^m)$. *For some constant* C, *depending on* m *and* N *only, then*

$$\|a_1*\cdots*a_N\|_{M^{p_0,q_0}}\leqslant C\left(\prod_{j=1}^{N_0}\|a_j\|_{M^{p_j,q_j}}\right)\left(\prod_{k=N_0+1}^{N}\|a_k\|_{L^{p_k}}\right), \tag{2.8}$$

for every $a_j\in M^{p_j,q_j}(\mathbf{R}^m)$ *when* $1\leqslant j\leqslant N_0$ *and* $a_k\in L^{p_k}(\mathbf{R}^m)$ *when* $N_0+1\leqslant k\leqslant N$.

Moreover, if in addition $q_0\leqslant\min(p_0,p_0')$, *then the same conclusion holds when* M^{p_0,q_0} *and its norm are replaced by* L^{p_0} *and its norm.*

In the case that m *is even, then the conclusion above and estimate* (2.8) *still holds, after* $L^{p_k}(\mathbf{R}^m)$ *is replaced by* $s_{t_k,p_k}(\mathbf{R}^m)$, *for one or more* k *in* $\{0,N_0+1,\dots,N\}$ *and* $t_k\in\mathbf{R}$ *for every* k.

In the case $N_0\geqslant N$, then (2.7) is interpreted as $M^{p_1,q_1}\times\cdots\times M^{p_N,q_N}$, and then Theorem 2.6 essentially agrees with Theorem 2.5.

Remark 2.7. We note that Theorems 2.4 and 2.5 are still true if q_j, $1\leqslant j\leqslant N$, are replaced by smaller numbers. In particular, Theorems 2.4 and 2.5 remain valid if the conditions on p_j and q_j are replaced by

$$1/p_1+\cdots+1/p_N=N-1+1/p_0\quad\text{and}\quad 1/q_1+\cdots+1/q_N=N-1+1/q_0.$$

Remark 2.8. It follows from (2.3) and similar arguments which lead to Theorem 2.5, that the multiplication $a_1\cdots a_N$ on $\mathscr{S}$ extends in a natural way to a continuous mapping from $M^{q_1,p_1}\times\cdots\times M^{q_N,p_N}$ to M^{q_0,p_0}, provided $p_j,q_j\in[1,\infty]$ satisfy (2.1).

In particular, $M^{p,q}=S_0^0\cdot M^{p,q}=M^{\infty,1}\cdot M^{p,q}$, since $1\in S_0^0\subset M^{\infty,1}$.

In the same way, the convolution $a_1*\cdots*a_N$ on $\mathscr{S}(\mathbf{R}^m)$ extends to a continuous N-linear map from $M_{\mathscr{F}}^{p_1,q_1}\times\cdots\times M_{\mathscr{F}}^{p_N,q_N}$ to $M_{\mathscr{F}}^{p_0,q_0}$.

This result can also be obtained from Theorem 3 in [9], by choosing appropriate L^p-spaces and $\mathscr{F}L^p$-spaces for the involving Banach spaces.

Remark 2.9. A time after that the present paper was submitted, the paper [5] appeared. Here Cordero and Gröchenig give an independent but similar proof of estimate (2.6) in the case of certain types of weighted modulation spaces. (Cf. [5, Proposition 2.4].)

Remark 2.10. In Sections 2.2 and 2.3 in [32] or in Sections 2 and 3 in [35], other results of Young-type for L^p-spaces, s_p^w-spaces and dilated s_p^w-spaces, are presented.

Remark 2.11. We note that $C'_B(\mathbf{R}^m)$, the set of Borel measures on $\mathbf{R}^m$ with finite mass, is contained in $M^{1,\infty}(\mathbf{R}^m)$. In fact, if $\mu \in C'_B(\mathbf{R}^m)$, then

$$\int |\mathscr{F}(\mu\tau_x\chi)(\xi)|\,dx = \pi^{-m/2}\int \left| \int \chi(y-x)e^{-2i\langle y,\xi\rangle}\,d\mu(y)\right| dx$$
$$\leqslant \pi^{-m/2}||\chi||_{L^1}||\mu||,$$

where $||\mu||$ denotes the total mass for μ. The result follows now by taking the supremum of the left-hand side with respect to the ξ-variable.

In particular, Theorem 2.5 shows that $(M^{1,\infty},*)$ is a commutative ring (with unit δ_0) which is a superring to $(C'_B,*)$. More generally, Theorem 2.5 shows that $M^{p,q}(\mathbf{R}^m)$ is an $M^{1,\infty}$-module under convolution, for every $p,q\in[1,\infty]$. The last result is also a consequence of Corollary 2.12 in [11].

3. Inclusions between modulation spaces and Besov spaces

In this section we apply the convolution results from the last section together with a few estimates in order to obtain inclusion relations between modulation spaces and Besov spaces. In some cases, the results are sharp and may not be improved, while in the other cases it seems to be an open question for the author, whether the results might be improved or not. We also note that Gröbner [17] and Okoudjou [26], have obtained inclusion relations between modulation spaces and Besov/Sobolev/Triebel spaces. In a few cases, our result agrees with Theorem F.4 in [17], while in the other cases we get strict improvements. In [26], the approach is based on certain methods in time–frequency analysis, and the results are also partially different and not always comparable to the results presented here.

We start by recalling some basic facts for the Besov spaces and Sobolev spaces. Assume that $p,q\in[1,\infty]$ and that $s\in\mathbf{R}$. Assume also that $\psi_0,\psi\in C_0^\infty(\mathbf{R}^m)$ are non-negative and satisfies $0\notin\operatorname{supp}\psi$, and $\sum_0^\infty \psi_k = 1$, where $\psi_k = \psi(\cdot/2^k)$ when $k\geqslant 1$. Then the Besov space $B_s^{p,q}(\mathbf{R}^m)$ is defined as the set of all $a\in\mathscr{S}'(\mathbf{R}^m)$ such that

$$||a||_{B_s^{p,q}} \equiv \left(\sum_{k=0}^\infty (2^{ks}||\psi_k(D)a||_{L^p})^q\right)^{1/q}$$

is finite. If $p=q$, then the notations B_s^p and $||\cdot||_{B_s^p}$ are used instead of $B_s^{p,p}$ and $||\cdot||_{B_s^{p,p}}$, respectively. We observe that (1.5) and Fourier's inversion formula imply that if $\varphi\in\mathscr{S}$, then

$$\varphi(D)a = (4\pi)^{-m/2}((\mathscr{F}\check{\varphi})(\cdot/2))*a. \tag{3.1}$$

Here we recall that $\check{\varphi}(x)=\varphi(-x)$. We also note that for any $p,q\in[1,\infty]$, $B_s^{p,q}(\mathbf{R}^m)$ is a Banach space which is independent of ψ_0 and ψ above, and that different choices of

ψ_0 and ψ give rise to equivalent norms. If the Besov norm is fix, then $a \mapsto \sup |(a,b)|$ is an equivalent norm to $\|\cdot\|_{B_s^{p,q}}$. Here the supremum should be taken over all $b \in B_{-s}^{p',q'}$ such that $\|b\|_{B_{-s}^{p',q'}} \leqslant 1$. If in addition $p, q < \infty$, then the dual space for $B_s^{p,q}$ is given by $B_{-s}^{p',q'}$.

The Sobolev space $H_s^p(\mathbf{R}^m)$ is defined as the set of all $a \in \mathscr{S}'(\mathbf{R}^m)$ such that $(1+|D|^2)^{s/2} a \in L^p(\mathbf{R}^m)$. We recall the usual inclusion relations between Besov spaces and Sobolev spaces. Assume that $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, $1 \leqslant p \leqslant \infty$ and $s_1 < s < s_2$. Then

$$\begin{aligned} &B_s^{p,q_1} \hookrightarrow B_s^{p,q_2}, \quad &&B_s^{p,\min(p,p')} \hookrightarrow H_s^p \hookrightarrow B_s^{p,\max(p,p')}, \\ &B_s^2 = H_s^2, &&B_{s_2}^{p,\infty} \hookrightarrow H_s^p \hookrightarrow B_{s_1}^{p,1}. \end{aligned} \tag{3.2}$$

In particular, for any inclusion relation involving Besov spaces, we obtain at the same time similar relations where the Besov spaces are replaced by Sobolev spaces. We refer to Chapter 6 in [1] for more facts concerning Besov and Sobolev spaces.

Some of our investigations are based on Minkowski's inequality, in a somewhat general form. Recall that for a $d\nu$-measurable function f with values in the Banach space B with norm $\|\cdot\|$, Minkowski's inequality asserts that $\|\int f\,d\nu\| \leqslant \int \|f\|\,d\nu$. In our applications, B is equal to $L^p(d\mu)$, for some $p \in [1,\infty]$, and Minkowski's inequality takes the form $(\int |\int f\,d\nu|^p\,d\mu)^{1/p} \leqslant \int (\int |f|^p\,d\mu)^{1/p}\,d\nu$.

Since the definition of Besov spaces are independent of the choice of ψ_0 and ψ, we assume from now on that ψ_0 and ψ are fixed.

We have now the following embedding result between modulation spaces and Besov spaces.

Theorem 3.1. *Assume that $p, q, p_j, q_j \in [1,\infty]$, $j = 1, 2$, satisfy $p_1 \leqslant p \leqslant p_2$ and $q_1 \leqslant q \leqslant q_2$, and assume that θ_1 and θ_2 are the same as in* (0.2). *Then*

$$B_{m\theta_1(p_1,q_1)}^{p_1,q_1}(\mathbf{R}^m) \hookrightarrow M^{p,q}(\mathbf{R}^m) \hookrightarrow B_{m\theta_2(p_2,q_2)}^{p_2,q_2}(\mathbf{R}^m),$$

and for some constant $C_m > 0$, depending on m only,

$$C_m^{-1}\|a\|_{B_{m\theta_2(p_2,q_2)}^{p_2,q_2}} \leqslant \|a\|_{M^{p,q}} \leqslant C_m \|a\|_{B_{m\theta_1(p_1,q_1)}^{p_1,q_1}}, \quad a \in \mathscr{S}'(\mathbf{R}^m). \tag{3.3}$$

In particular, (0.3) *holds.*

We note that Theorem 3.1 can be formulated with only one of the functions θ_1 and θ_2, since $\theta_2(p,q) = \theta_2(p',q) = -\theta_1(p',q') = -\theta_1(p,q')$.

Remark 3.2. In the proof of Theorem 3.1 we shall apply some (complex) interpolation techniques for Besov spaces $B_s^{p,q}$ and Modulation spaces $M^{p,q}$. In general, interpolation works out properly as long as $\mathscr{S}$ is dense in the spaces under consideration. In our situation, $\mathscr{S}$ is neither dense in $B_s^{p,q}$ nor in $M^{p,q}$, as soon as $p = \infty$ or $q = \infty$. We may however avoid this obstacle by considering the spaces

$\mathscr{B}_s^{p,q}$ and $\mathscr{M}^{p,q}$, the completions of $\mathscr{S}$ under the norms $\|\cdot\|_{B^{p,q}}$ and $\|\cdot\|_{M^{p,q}}$, respectively. In fact, it follows from Proposition 1.2, Corollary 2.3 in [8] and the analysis in Section 5.6 and Section 6.4 in [1], that the following are true:

(1) if $p, q < \infty$, then $\mathscr{B}_s^{p,q} = B_s^{p,q}$ and $\mathscr{M}^{p,q} = M^{p,q}$;
(2) if $p_j, q_j \in [1, \infty]$, $s_j \in \mathbf{R}$ for $j \in \{0, 1, 2\}$ and $0 < \theta < 1$ satisfy

$$\frac{1}{p_0} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_0} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2} \quad \text{and} \quad s_0 = \theta s_1 + (1-\theta)s_2,$$

then

$$(\mathscr{B}_{s_1}^{p_1,q_1}, \mathscr{B}_{s_2}^{p_2,q_2})_{[\theta]} = \mathscr{B}_{s_0}^{p_0,q_0} \quad \text{and} \quad (\mathscr{M}^{p_1,q_1}, \mathscr{M}^{p_2,q_2})_{[\theta]} = \mathscr{M}^{p_0,q_0}.$$

Remark 3.3. In [17], Gröbner considers embedding properties between different α-modulation spaces. Here it is proved that $B_{m/q}^{p,q} \hookrightarrow M^{p,q} \hookrightarrow B_{-m/q'}^{p,q}$ (cf. Theorem F.4). This result is also improved by interpolation, using the fact that $M^2 = B_0^2$. In this context, Theorem 3.1 is an improvement of these results.

Some parts of the proof of Theorem 3.1 are based on the following lemma which is a consequence of Theorem F.4 in [17]. (See Remark 3.3.) Here we give an alternative proof.

Lemma 3.4. *If $p \in [1, \infty]$, then $M^{p,1} \hookrightarrow B_0^{p,1}$ and $B_0^{p,\infty} \hookrightarrow M^{p,\infty}$.*

Proof. By duality, it suffices to prove the second inclusion. Assume that $a \in B^{p,\infty}$, $\chi \in C_0^\infty(\mathbf{R}^m) \backslash 0$ is real-valued and $N \geqslant 1$ is a large integer. Then for every $\xi \in \mathbf{R}^m$, there is a k such that $\psi_k + \dots + \psi_{N+k} = 1$ in the support of $\tau_\xi \chi$, provided N is chosen large enough. This gives

$$H_{a,p,\hat{\chi}}(\xi) = \left(\int |\mathscr{F}(\hat{a}\tau_\xi \chi)(x)|^p \, dx \right)^{1/p} \leqslant \sum_{j=k}^{N+k} I_j(\xi), \tag{3.4}$$

where

$$\begin{aligned} I_j(\xi) &= \left(\int |\mathscr{F}(\psi_j \hat{a} \tau_\xi \chi)(x)|^p \, dx \right)^{1/p} \\ &= \pi^{-m/2} \left(\int |(\psi_j \hat{a}, e^{2i\langle x, \cdot \rangle} \tau_\xi \chi)|^p \, dx \right)^{1/p}. \end{aligned}$$

We have to estimate $I_j(\xi)$. By Parseval's formula and Young's inequality we get

$$\begin{aligned} I_j(\xi) &= \pi^{-m/2} \left(\int |(\mathscr{F}(\psi_j \hat{a}), e^{2i\langle \cdot, \xi \rangle} \tau_{-x} \hat{\chi})|^p \, dx \right)^{1/p} \\ &\leqslant \| \, |\mathscr{F}(\psi_j \hat{a})| * |\hat{\chi}| \, \|_{L^p} \leqslant \|\mathscr{F}(\psi_j \hat{a})\|_{L^p} \|\hat{\chi}\|_{L^1} \leqslant C_\chi \|a\|_{B_0^{p,\infty}}, \end{aligned}$$

for some constant C_χ independent of j. Since N in (3.4) is independent of j, it follows that $H_{a,p,\hat{\chi}}(\xi) \leqslant C||a||_{B_0^{p,\infty}}$. By applying the L^∞-norm with respect to the ξ-variable on the last inequality, we get $||a||_{M^{p,\infty}} \leqslant C||a||_{B_0^{p,\infty}}$. The proof is complete. $\square$

Proof of Theorem 3.1. In view of Proposition 1.2, we may assume that $p_1 = p_2 = p$, and $q_1 = q_2 = q$. By duality and (3.2), it also suffices to prove the inclusion

$$M^{p,q}(\mathbf{R}^m) \hookrightarrow B^{p,q}_{m\theta_2(p,q)}(\mathbf{R}^m). \tag{3.5}$$

Assume that $q_0 \in [1, \infty]$ satisfies $1/q_0 + 1/q = 1/\min(p, p')$, and that $a \in M^{p,q}(\mathbf{R}^m)$. Then Lemma 1.8 and Theorem 2.6 show that

$$\begin{aligned} ||e^{-\lambda^2|D|^2/4}a||_{L^p} &= \pi^{-m/2}\lambda^{-m}||u_{1/\lambda^2} * a||_{L^p} \leqslant C\lambda^{-m}||u_{1/\lambda^2}||_{M^{1,q_0}}||a||_{M^{p,q}} \\ &\leqslant C_1\lambda^{-m/q_0}(1+\lambda^2)^{m/(2q_0)}||a||_{M^{p,q}}, \end{aligned}$$

for some constants C and C_1. Hence for some constant C,

$$||e^{-\lambda^2|D|^2}a||_{L^p} \leqslant C\lambda^{-m/q_0}||a||_{M^{p,q}} \quad \text{when } 0 < \lambda \leqslant 1. \tag{3.6}$$

Next, we consider ψ and ψ_0 in the definition of Besov spaces. Then $\phi(x) = \psi(x)e^{|x|^2} \in C_0^\infty(\mathbf{R}^m)$, which implies that $\{\lambda^{-m}\hat{\phi}(\cdot/\lambda)\}_{0<\lambda\leqslant 1}$ is a bounded set in L^1. A combination of this fact with Young's inequality gives

$$\begin{aligned} ||\psi(\lambda D)a||_{L^p} &= ||\phi(\lambda D)e^{-\lambda^2|D|^2}a||_{L^p} \\ &= (4\pi)^{-m/2}\lambda^{-m}||(\hat{\phi}(\cdot/(2\lambda)) * (e^{-\lambda^2|D^2}a)||_{L^p} \leqslant C||e^{-\lambda^2|D|^2}a||_{L^p}, \end{aligned}$$

for some constant C. From the last estimate together with (3.6), it follows that

$$||\psi(\lambda D)a||_{L^p} \leqslant C\lambda^{-m/q_0}||a||_{M^{p,q}}, \quad \text{when } 0 < \lambda \leqslant 1, \tag{3.7}$$

for some constant C. Hence substituting λ with 2^{-k}, we land on

$$2^{-mk/q_0}||\psi_k(D)a||_{L^p} \leqslant C||a||_{M^{p,q}}, \quad k \geqslant 0,$$

where

$$1/q + 1/q_0 = 1/\min(p, p'). \tag{3.8}$$

Now let $q = \infty$ in (3.8). Then $q_0 = \min(p, p')$ and

$$2^{-mk/\min(p,p')}||\psi_k(D)a||_{L^p} \leqslant C||a||_{M^{p,\infty}}.$$

By taking supremum over k we get $M^{p,\infty} \hookrightarrow B^{p,\infty}_{-m/\min(p,p')}$, which gives (3.5) when $1 \leqslant p \leqslant \infty$, $q = \infty$.

If instead $q=1$, then (3.5) follows from Proposition 1.2(1) and Lemma 3.4.

Next we consider the case $p=1$, $1\leqslant q\leqslant\infty$. Since $M^{1,2}\hookrightarrow M^2=B_0^2$, it follows from the above and Lemma 3.4 that

$$M^{1,2}\hookrightarrow B_0^2,\quad M^{1,\infty}\hookrightarrow B_{-m}^{1,\infty},\quad M^1\hookrightarrow B_0^1. \tag{3.9}$$

By interpolation of the last two inclusions in (3.9) we get (3.5) in the case $p=1$, $1\leqslant q\leqslant\infty$.

In order to prove (3.5) in case $p=\infty$, $1\leqslant q\leqslant\infty$, it suffices to prove the dual result

$$B_{m\theta_1(p,q)}^{p_1,q}\hookrightarrow M^{p,q} \tag{3.5$'$}$$

when $p=1$, $1\leqslant q\leqslant\infty$. It was proved above that $M^\infty\hookrightarrow B_{-m}^\infty$, which implies that $B_m^1\hookrightarrow M^1$ by duality. This gives together with Lemma 3.4 and interpolation that $B_{m/q}^{1,q}\hookrightarrow M^{1,q}$, and we conclude that (3.5′) holds for $p=1$.

We have therefore proved (3.5) in the case $p\in\{1,\infty\}$, $1\leqslant q\leqslant\infty$, and in the case $1\leqslant p\leqslant\infty$, $q\in\{1,\infty\}$. For general p and q we obtain now (3.5) from these results, the fact that $M^2=B_0^2$, and by interpolation. The proof is complete. □

Corollary 3.5. *Assume that $t\in\mathbf{R}$ and that $p\in[1,\infty]$. Then the following inclusions hold*:

$$\begin{aligned}B_{2m|1-2/p|}^{p,\min(p,p')}(\mathbf{R}^{2m})&\hookrightarrow M^{p,\min(p,p')}(\mathbf{R}^{2m})\hookrightarrow s_{t,p}(\mathbf{R}^{2m})\\&\hookrightarrow M^{p,\max(p,p')}(\mathbf{R}^{2m})\hookrightarrow B_{-2m|1-2/p|}^{p,\max(p,p')}(\mathbf{R}^{2m}).\end{aligned}$$

In particular (0.3) is obtained from these relations together with (3.1).

Proof. The result is an immediate consequence of Proposition 1.6 and Theorem 3.1. □

Remark 3.6. We note that Corollary 3.5 does not contain Theorem 2.6 in [35] or the embedding results in [2,4], which imply that $B_m^{\infty,1}(\mathbf{R}^{2m})\subset s_{t,\infty}(\mathbf{R}^{2m})$ when $t=0$ or $t=1/2$. It does however improve the weaker result Theorem 2.6′ in [35] (the same as Theorem 2.2.7 in [32]), which asserts that if $p\in[1,\infty]$, then $B_{2m|1-2/p|}^{p,\min(p,p')}\hookrightarrow s_p^w\hookrightarrow B_{-2m|1-2/p|}^{p,\max(p,p')}$.

Remark 3.7. It follows from Theorem 3.1 that S_0^0 is continuously embedded in $B_0^{\infty,1}$, since $S_0^0\hookrightarrow M^{\infty,1}$.

Remark 3.8. The proofs of Theorem 3.1 and Corollary 3.5 also work with no difference for inclusion relations between $M^{p,q}(\mathbf{R}^m)$, and the modified Besov spaces $B_{s_1,\dots,s_m}^{p,q}(\mathbf{R}^m)$ equipped with corresponding modified Besov norm, discussed in [2,4]. It follows that the conclusions in Theorem 3.1 are true when $B^{p_j,q_j}_{ms_j}(\mathbf{R}^m)$ together with their norms are replaced by $B^{p_j,q_j}_{s_j,\dots,s_j}(\mathbf{R}^m)$ and their norms, for $j = 1, 2$.

Remark 3.9. In [36], one presents alternative proofs of certain parts of Theorem 3.1, based on more straightforward computations.

Remark 3.10. Theorem 3.1 also gives a link about differences between $s^w_1(\mathbf{R}^{2m})$ and $M^1(\mathbf{R}^{2m})$, since $M^1(\mathbf{R}^{2m}) \hookrightarrow B^1_0(\mathbf{R}^{2m})$ and $s^w_1(\mathbf{R}^{2m}) \hookrightarrow B^{1,\infty}_{-m}(\mathbf{R}^{2m})$ but $s^w_1(\mathbf{R}^{2m}) \nsubseteq B^{1,\infty}_s(\mathbf{R}^{2m})$ when $s > -m$ (cf. Remark 2.10 in [35]).

Remark 3.11. It follows from Theorem 3.1 that $B^p_{ms_1} \hookrightarrow M^p$ when $s_1 = 2/p - 1$ and $1 \leqslant p \leqslant 2$, and that $M^p \hookrightarrow B^p_{ms_2}$ when $s_2 = 2/p - 1$ and $2 \leqslant p \leqslant \infty$. On the other hand, if s_1 is replaced by a strictly smaller value or s_2 by a strictly larger value, then the corresponding inclusion fails to hold.

In fact, by duality, (3.1) and the closed graph theorem it suffices to prove that if $\varepsilon > 0$ and $1 \leqslant p \leqslant 2$, then for some bounded sequence $\{a_\lambda\}_{\lambda \geqslant 1}$ in $H^p_{m(2/p-1)-\varepsilon}$ it follows that $\|a_\lambda\|_{M^p} \to \infty$ as $\lambda \to \infty$. Let us consider $u_\lambda = e^{-\lambda|x|^2}$. Lemma 1.8 gives for some constant $C > 0$ that $C^{-1}\lambda^{-m/2p'} \leqslant \|u_\lambda\|_{M^p} \leqslant C\lambda^{-m/2p'}$ as $\lambda \geqslant 1$. On the other hand, by Remark 2.10 in [35] it follows that $\|u_\lambda\|_{H^p_{m(2/p-1)-\varepsilon}} \leqslant C\lambda^{-m/2p'-\varepsilon/4}$, for some constant C. The result follows now if we let $a_\lambda = \lambda^{m/2p'+\varepsilon/4}u_\lambda$.

4. Applications to pseudo-differential operators and Toeplitz operators

In this section we discuss continuity properties for pseudo-differential operators in the context of modulation spaces. We generalize Theorem 14.5.2 in [19] and give sufficient conditions on $p_1, p_2, q_1, q_2 \in [1, \infty]$ such that the pseudo-differential operator $a_t(x, D)$ is continuous from $M^{p_1,q_1}(\mathbf{R}^m)$ to $M^{p_2,q_2}(\mathbf{R}^m)$, when $a \in M^{p,q}(\mathbf{R}^{2m})$, $p, q \in [1, \infty]$, and $t \in \mathbf{R}$.

We also give complementary results to Proposition 1.7, and prove that if $p \in [1, \infty]$ and $q > 2$, then there is an element $a \in M^{p,q}(\mathbf{R}^{2m})$ such that $a_t(x, D)$ is not continuous on L^2.

The investigations are based on certain relations between the ambiguity function and the Weyl quantization together with some continuity properties for the ambiguity function in background of modulation spaces.

We also apply the convolution results from Section 2 to Toeplitz operators, and prove for example that if $h_0 \in M^1(\mathbf{R}^m)$, then the map $a \mapsto \mathrm{Tp}_{h_0}(a)$ from $\mathscr{S}$ to $\mathscr{I}_\infty$ uniquely extends in a continuous way to a map from $M^{p,q}$ to $\mathscr{I}_p$, for every $p, q \in [1, \infty]$.

We start by discussing basic relations between the Weyl quantization and the ambiguity function. If $a \in \mathscr{S}'(\mathbf{R}^{2m})$ and $f, g \in \mathscr{S}(\mathbf{R}^m)$, then it follows by straightforward computations that

$$\langle a^w(x,D)f, g\rangle = (2\pi)^{-m/2}\langle a, W_{f,\check{g}}\rangle$$

and

$$(a^w(x,D)f, g) = (2\pi)^{-m/2}(a, W_{\check{g},\bar{f}}). \tag{4.1}$$

In the case $a = W_{f_1,f_2}$ where $f_1,f_2 \in \mathscr{S}'(\mathbf{R}^m)$, then it follows from (4.1) that

$$a^w(x,D)f(x) = (2\pi)^{-m/2}\langle f, f_2\rangle f_1(-x). \tag{4.2}$$

Here we recall that the map $(f_1,f_2) \mapsto W_{f_1,f_2}$ is continuous from $\mathscr{S}'(\mathbf{R}^m) \times \mathscr{S}'(\mathbf{R}^m)$ to $\mathscr{S}'(\mathbf{R}^{2m})$ which restricts to a continuous mapping from $\mathscr{S}(\mathbf{R}^m) \times \mathscr{S}(\mathbf{R}^m)$ to $\mathscr{S}(\mathbf{R}^{2m})$, and from $L^2(\mathbf{R}^m) \times L^2(\mathbf{R}^m)$ to $L^2(\mathbf{R}^{2m})$. (See also [16,32] or [35].) We also have that $W_{f_1,f_2} \in M^1$ when $f_1,f_2 \in M^1(\mathbf{R}^m)$ (cf. Subsection 3.2.4 in [15] or Proposition 12.1.2 in [19]). In the following, we generalize this result to more general modulation spaces.

Theorem 4.1. *Assume that $p_j, q_j, p, q \in [1, \infty]$ such that $p \leqslant p_j, q_j \leqslant q$, for $j = 1, 2$, and that*

$$1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p + 1/q.$$

Then the following are true:

(1) *the map $(f_1,f_2) \mapsto W_{f_1,f_2}$ from $\mathscr{S}'(\mathbf{R}^m) \times \mathscr{S}'(\mathbf{R}^m)$ to $\mathscr{S}'(\mathbf{R}^{2m})$ restricts to a continuous map from $M^{p_1,q_1}(\mathbf{R}^m) \times M^{p_2,q_2}(\mathbf{R}^m)$ to $M^{p,q}(\mathbf{R}^{2m})$. If $\psi = W_{\chi_1,\chi_2}$ where $\chi_j \in \mathscr{S}(\mathbf{R}^m)$, then*

$$\begin{aligned} &\|W_{f_1,f_2}\|_{M^{p,q,\bar{\psi}}} \leqslant 2^{(1/p-1/q)m}\|f_1\|_{M^{p_1,q_1,\bar{\chi}_1}}\|f_2\|_{M^{p_2,q_2,\bar{\chi}_2}}, \\ &\text{where} \quad f_1,f_2 \in \mathscr{S}'(\mathbf{R}^m). \end{aligned} \tag{4.3}$$

In particular, if $f_1 \in M^p(\mathbf{R}^m)$ and $f_2 \in M^q(\mathbf{R}^m)$, or $f_1 \in M^{p,q}(\mathbf{R}^m)$ and $f_2 \in M^{q,p}(\mathbf{R}^m)$, then $W_{f_1,f_2} \in M^{p,q}(\mathbf{R}^{2m})$;

(2) *if in addition $p = q$, then $W_{f_1,f_2} \in M^p(\mathbf{R}^{2m})$ if and only if $f_1 \in M^p(\mathbf{R}^m)$ and $f_2 \in M^p(\mathbf{R}^m)$, and equality is attained in* (4.3).

Before the proof we recall the homeomorphism A on $\mathscr{S}(\mathbf{R}^{2m})$ which is defined as

$$(Aa)(x,y) = (2\pi)^{-m/2}\int a((y-x)/2, \xi)e^{-i\langle x+y,\xi\rangle}\, d\xi.$$

Then A extends to a homeomorphism on $\mathscr{S}'(\mathbf{R}^{2m})$ which is unitary on L^2, and $A(W_{f_1,f_2}) = f_1 \otimes f_2$ for every $f_1,f_2 \in \mathscr{S}'(\mathbf{R}^m)$. (Cf. [16] or [32–35].)

Proof. The result follows if we prove that (4.3) holds, with equality for $p = q$. Set $U = A\psi = \chi_1 \otimes \chi_2$, $X = (x, \xi) \in \mathbf{R}^{2m}$ and $Y = (y, \eta) \in \mathbf{R}^{2m}$, and let $\langle X, Y \rangle = \langle x, \eta \rangle + \langle y, \xi \rangle$. If $x' = y - x$, $y' = x + y$, $\xi' = -(\xi - \eta)$ and $\eta' = -(\xi + \eta)$, then it follows by a straightforward applications of Fourier's inversion formula that

$$\begin{aligned} A(e^{2i\langle Y, \cdot \rangle}\tau_X\psi)(z, w) &= e^{i(\langle w-z, \eta \rangle - \langle z+w-2y, \xi \rangle)} U(z - x', w - y') \\ &= e^{2i\langle y, \xi \rangle} e^{i\langle z, \eta' \rangle} e^{i\langle w, \xi' \rangle} \chi_1(z - x')\chi_2(w - y'). \end{aligned}$$

This gives together with the fact that A is unitary on $L^2(\mathbf{R}^m)$ that

$$\begin{aligned} \mathscr{F}(W_{f_1,f_2}\overline{\tau_X\psi})(Y) &= \pi^{-m}(W_{f_1,f_2}, e^{2i\langle Y \cdot \rangle}\tau_X\psi) \\ &= \pi^{-m}(A(W_{f_1,f_2}), A(e^{2i\langle Y, \cdot \rangle}\tau_X\psi)) \\ &= e^{-2i\langle y, \xi \rangle}(f_1, e^{i\langle \cdot, \eta' \rangle}\tau_{x'}\chi_1)(f_2, e^{i\langle \cdot, \xi' \rangle}\tau_{y'}\chi_2) \\ &= e^{-2i\langle y, \xi \rangle}\mathscr{F}(f_1\tau_{x'}\bar{\chi}_1)(\eta'/2)\mathscr{F}(f_2\tau_{y'}\bar{\chi}_2)(\xi'/2). \end{aligned}$$

Hence

$$|\mathscr{F}(W_{f_1,f_2}\tau_X\bar{\psi})(Y)| = |\mathscr{F}(f_1\tau_{x'}\bar{\chi}_1)(\eta'/2)||\mathscr{F}(f_2\,\tau_{y'}\bar{\chi}_2)(\xi'/2)|. \tag{4.4}$$

Applying the L^p-norm gives

$$||W_{f_1,f_2}||_{M^{p,p,\bar{\psi}}} = ||f_1||_{M^{p,p,\bar{\chi}_1}}||f_2||_{M^{p,p,\bar{\chi}_2}}, \tag{4.3$'$}$$

which proves (2).

In order to prove (1) we consider (4.4) again. For $j = 1, 2$ we set $G_j(x, \xi) = |\mathscr{F}(f_j\tau_x\bar{\chi}_j)(\xi)|^p$. Then (4.4) gives

$$||W_{f_1,f_2}||_{M^{p,q,\bar{\psi}}} = 2^{m/p}\left(\int F(-\eta)\,d\eta\right)^{1/q}, \tag{4.5}$$

where

$$\begin{aligned} F(\eta) &= \int \left(\int\int G_1(x', -\xi + \eta/2)G_2(y', \xi + \eta/2)\,dx\,d\xi\right)^{q/p} dy \\ &= 2^{-m}\int \left(\int\int G_1(y - x, \eta - \xi)G_2(x, \xi)\,dx\,d\xi\right)^{q/p} dy. \end{aligned}$$

Since $r = q/p \geqslant 1$, it follows from Minkowski's inequality that

$$F(\eta) \leqslant 2^{-m}\left(\int\left(\int\left(\int G_1(y - x, \eta - \xi)G_2(x, \xi)\,dx\right)^{q/p} dy\right)^{p/q} d\xi\right)^{q/p}.$$

Assume now that $r_1, r_2 \in [1, \infty]$ are chosen such that $1/r_1 + 1/r_2 = 1 + p/q$. Then Young's inequality gives

$$F(\eta) \leqslant 2^{-m} \left(\int (||G_1(\cdot, \eta - \xi)||_{L^{r_1}} ||G_2(\cdot, \xi)||_{L^{r_2}}) \, d\xi \right)^{q/p}. \tag{4.6}$$

Next, we choose $s_1, s_2 \in [1, \infty]$ such that $1/s_1 + 1/s_2 = 1 + p/q$. By inserting (4.6) into (4.5) and applying Young's inequality again, we obtain

$$||W_{f_1,f_2}||_{M^{p,q,\tilde{\psi}}} \leqslant 2^{(1/p-1/q)m} \alpha_1 \alpha_2,$$

where $\alpha_j = (\int ||H_j(\cdot, \xi)||_{L^{r_j}}^{s_j} \, d\xi)^{1/(ps_j)}$, for $j = 1, 2$. Since $H_j(x, \xi) = |\mathscr{F}(f_j \tau_x \bar{\chi}_j)(\xi)|^p$, the result follows by letting $p_j = pr_j$ and $q_j = ps_j$. The proof is complete. $\square$

We can now prove the following complementary result to Proposition 1.6.

Corollary 4.2. *Assume that $p, q \in [1, \infty]$ and that $t \in \mathbf{R}$. Then the following are true:*

(1) *if $q > 2$, then $M^{p,q}(\mathbf{R}^{2m}) \nsubseteq s_{t,\infty}(\mathbf{R}^{2m})$;*
(2) *if $q < 2$, then $s_{t,1}(\mathbf{R}^{2m}) \nsubseteq M^{p,q}(\mathbf{R}^{2m})$.*

Proof. By duality it suffices to prove (1). From Proposition 1.1 and Remark 1.5, we may assume that $t = 1/2$ and that $p = 1$. Let $a = W_{f_1,\bar{f}_2}$, where $f_1 \in M^{q,1}(\mathbf{R}^m) \backslash L^2(\mathbf{R}^m)$ and $0 \neq f_2 \in M^{1,q}(\mathbf{R}^m)$. These choices are possible in view of Proposition 1.7. By Theorem 4.1 we have that $a \in M^{1,q}(\mathbf{R}^{2m})$. On the other hand, if $f \in \mathscr{S}(\mathbf{R}^m)$, then $(a^w(x, D)f)(x) = (2\pi)^{-m/2}(f, f_2) f_1(-x)$, and it is clear that the right-hand side is not an L^2-function when f is chosen such that $(f, f_2) \neq 0$. This proves that $a^w(x, D) \notin \mathscr{I}_\infty$, which is the same as $a \notin s_\infty^w$. The proof is complete. $\square$

We shall next apply Theorem 4.1 to pseudo-differential calculus. For admissible $a \in \mathscr{S}'(\mathbf{R}^{2m})$ and $f \in \mathscr{S}'(\mathbf{R}^m)$, $a^w(x, D)f(x)$ is from now on defined by (4.1), provided that the right-hand sides are well defined for every $g \in \mathscr{S}(\mathbf{R}^m)$. The following result generalizes Theorem 14.5.2 in [19] and Theorem 1.1 in [20].

Theorem 4.3. *Assume that $t \in \mathbf{R}$ and that $p_j, q_j \in [1, \infty]$, when $j = 0, 1, 2$, satisfy $q_0 \leqslant p_2, q_2 \leqslant p_0$ and*

$$1/p_1 - 1/p_2 = 1/q_1 - 1/q_2 = 1 - 1/p_0 - 1/q_0,$$

and that $a \in M^{p_0,q_0}(\mathbf{R}^{2m})$. Then the map $a_t(x, D)$ from $\mathscr{S}(\mathbf{R}^m)$ to $\mathscr{S}'(\mathbf{R}^m)$ extends uniquely to a continuous mapping from $M^{p_1,q_1}(\mathbf{R}^m)$ to $M^{p_2,q_2}(\mathbf{R}^m)$.

Proof. In view of Remark 1.5, we may assume that $t = 1/2$. The conditions on p_j and q_j imply that

$$p_0' \leqslant p_1, q_1, p_2', q_2' \leqslant q_0', \quad 1/p_1 + 1/p_2' = 1/q_1 + 1/q_2' = 1/p_0' + 1/q_0'.$$

Hence it follows from Theorem 4.1 that for some constant $C > 0$ we have that

$$||W_{\check{g},\bar{f}}||_{M^{p_0',q_0'}} \leqslant C||f||_{M^{p_1,q_1}}||g||_{M^{p_2',q_2'}}$$

when $f \in M^{p_1,q_1}(\mathbf{R}^m)$ and $g \in M^{p_2',q_2'}(\mathbf{R}^m)$. By (4.1) we get for some constant $C > 0$ and $C' > 0$ that

$$\begin{aligned} |(a^w(x,D)f,g)| &= (2\pi)^{-m/2}|(a, W_{\check{g},\bar{f}})| \\ &\leqslant C||a||_{M^{p_0,q_0}}||W_{\check{g},\bar{f}}||_{M^{p_0',q_0'}} \leqslant C'||a||_{M^{p_0,q_0}}||f||_{M^{p_1,q_1}}||g||_{M^{p_2',q_2'}}. \end{aligned}$$

The result follows now by duality. The proof is complete. □

Remark 4.4. Independently by the author, Theorem 4.3 was proved in a somewhat different way by Gröchenig and Heil [21].

We shall next discuss Toeplitz operators. Assume that $h_0 \in L^2(\mathbf{R}^m)$ is a unit vector which is fixed. Then it follows from the introduction that for any admissible $a \in \mathscr{S}'(\mathbf{R}^{2m})$ we have

$$\mathrm{Tp}_{h_0}(a) = (a * u_{h_0})^w(x,D), \tag{4.7}$$

where $u_{h_0} = (2\pi)^{m/2} W_{\tilde{h}_0,h_0}$. For future reference we also note that $u_{h_0} \in s_1^w(\mathbf{R}^{2m})$. (Cf. Section 1.4 in [32] or Section 1 in [33–35].)

We shall now discuss sufficient conditions for the symbol a, in order to $\mathrm{Tp}_{h_0}(a)$ should belong to $\mathscr{I}_p$ for some $p \in [1, \infty]$. First, we note that Remark 4.7 in [35] gives that $\mathrm{Tp}_{h_0}(a)$ makes sense as an element in $\mathscr{I}_\infty$ when $a(\sqrt{2}\cdot) \in s_\infty^w$. More generally, the following result is an immediate consequence of Theorem 2.3 and Theorem 3.2 in [35], and (4.7).

Proposition 4.5. *Assume that $h_0 \in L^2(\mathbf{R}^m)$ and that $p \in [1, \infty]$, and let T be the map $a \mapsto \mathrm{Tp}_{h_0}(a)$ from $\mathscr{S}(\mathbf{R}^{2m})$ to $\mathscr{I}_\infty$. Then the following are true*:

(1) *T extends uniquely to a continuous mapping from $L^p(\mathbf{R}^{2m})$ to $\mathscr{I}_p$. If in addition $u_{h_0} \in L^1$, then it extends to a continuous mapping from $s_p^w(\mathbf{R}^{2m})$ to $\mathscr{I}_p$;*

(2) *T extends uniquely to a continuous mapping from the set $\{a \in \mathscr{S}'(\mathbf{R}^{2m}); a(\sqrt{2}\cdot) \in s_p^w(\mathbf{R}^{2m})\}$ to $\mathscr{I}_p$.*

We shall now consider the case when the symbols belong to certain modulation spaces, and we also remove the condition that $h_0 \in L^2$, and define the Toeplitz operator $\mathrm{Tp}_{h_0}(a)$ by (4.7), for admissible $h_0 \in \mathscr{S}'(\mathbf{R}^m)$ and $a \in \mathscr{S}'(\mathbf{R}^{2m})$. We have then the following.

Theorem 4.6. *Assume that* $r, p_j, q_j \in [1, \infty]$, $j \in \{0, 1, 2\}$, *satisfy* $p_1 \leqslant q_1$,

$$1/p_1 + 1/p_2 = 1 + 1/p_0, \quad 1/q_1 + 1/q_2 = 1/q_0, \quad \textit{and} \quad 1/p_1 + 1/q_1 = 2/r.$$

Assume also that $h_0 \in M^r(\mathbf{R}^m)$. *Then the map* $a \mapsto \mathrm{Tp}_{h_0}(a)$ *from* $\mathscr{S}(\mathbf{R}^{2m})$ *to* $\mathscr{I}_\infty$ *extends to a continuous mapping from* $M^{p_2,q_2}(\mathbf{R}^{2m})$ *to* $\mathrm{Op}(M^{p_0,q_0})$.

Proof. It follows that $p_1 \leqslant r \leqslant q_1$. By Theorem 4.1 we have that $u_{h_0} \in M^{p_1,q_1}$. Hence for every $a \in M^{p_2,q_2}(\mathbf{R}^{2m})$ it follows from Proposition 1.2 and Theorem 2.5 that

$$a * u_{h_0} \in M^{p_2,q_2} * M^{p_1,q_1} \subset M^{p_0,q_0}.$$

The result now follows from (4.7) and Remark 1.5. The proof is complete. □

Corollary 4.7. *Assume that* $t \in \mathbf{R}$, *that* $h_0 \in M^1(\mathbf{R}^m)$ *and that* $p, q \in [1, \infty]$. *Then the map* $a \mapsto \mathrm{Tp}_{h_0}(a)$ *from* $\mathscr{S}(\mathbf{R}^{2m})$ *to* $\mathscr{I}_\infty$ *extends uniquely to a continuous mapping from* $M^{p,q}(\mathbf{R}^{2m})$ *to* $\mathrm{Op}(M^{p,1})$.

Note that $\mathrm{Op}(M^{p,1})$ is a subset of $\mathscr{I}_p$ in view of Propositions 1.2 and 1.7.

Corollary 4.8. *Assume that* $h_0 \in L^2(\mathbf{R}^m)$, *and* $p, q \in [1, \infty]$. *Then the map* $a \mapsto \mathrm{Tp}_{h_0}(a)$ *from* $\mathscr{S}(\mathbf{R}^{2m})$ *to* $\mathscr{I}_\infty$ *extends to a continuous mapping from* $M^{p,q}(\mathbf{R}^{2m})$ *to* $\mathrm{Op}(M^{p,q})$. *In particular*, $\mathrm{Tp}_{h_0}(a)$ *is continuous from* $M^1(\mathbf{R}^m)$ *to* $M^\infty(\mathbf{R}^m)$ *when* $a \in M^\infty(\mathbf{R}^{2m})$.

Proof. By taking into account that $M^2 = L^2$, the first part follows by choosing $r = 2$, $p_1 = 1$ and $q_1 = \infty$ in Theorem 4.6. If in addition $p = q = \infty$, then the last part follows from Theorem 4.3. □

Remark 4.9. He and Wong proved already in [24] that the first part of (1) in Proposition 4.5 holds.

Remark 4.10. In [5], related results comparing to Theorems 4.1 and 4.6 are obtained. (See also Remark 2.9.)

We shall finally discuss pseudo-differential operators with symbols in S_0^0. It was remarked already in the introduction that such operators are continuous on L^2. In the following, we discuss some further continuity properties for such operators.

Theorem 4.11. *Assume that $a \in S_0^0(\mathbf{R}^{2m})$. Then the following are true:*

(1) *if $t \in \mathbf{R}$, then $a_t(x,D)$ is continuous on $S_0^0(\mathbf{R}^m)$;*
(2) *if $h_0 \in L^2(\mathbf{R}^m)$, then $\mathrm{Tp}_{h_0}(a) \in \mathrm{Op}(S_0^0)$.*

Lemma 4.12. *Assume that $a \in S_0^0$, that $\phi \in C_0^\infty$ such that $\phi(0) = 1$, and set $a_j = a\,\phi(\cdot/j)$, for every $j \geqslant 1$. Then $\partial^\alpha a_j \to \partial^\alpha a$ narrowly, for every α.*

Proof. The result is an immediate consequence of the first part of the proof of Proposition 2.3 in [34]. $\square$

Proof of Theorem 4.11. We may assume that $t = 1/2$. Since $S_0^0 \subset M^{\infty,1}$, it follows from Theorem 4.3 that $a^w(x,D) : S_0^0 \to M^{\infty,1}$ is continuous. We claim that if $f \in S_0^0(\mathbf{R}^m)$ and $v = \mathrm{Op}^w(a)f$, then

$$D_j v = \mathrm{Op}^w(D_j a)f + \mathrm{Op}^w(a)(D_j f). \tag{4.8}$$

In fact, assume that $\varphi \in \mathscr{S}(\mathbf{R}^m)$. Then (4.1) gives with $\psi = D_j\varphi$ that $\langle D_j v, \varphi \rangle = -\langle v, D_j\varphi \rangle = -(2\pi)^{-m/2}\langle a, W_{f,\check{\psi}} \rangle$. Hence (4.8) follows from (4.1), Lemma 1.9(3) and Lemma 4.12 since $W_{f,\check{\psi}} = D_j W_{f,\check{\varphi}} - W_{D_j f,\check{\varphi}}$.

It follows from (4.8) that $D_j v \in M^{\infty,1}$, since $D_j f \in S_0^0$ and $D_j a \in S_0^0$. Hence v and $D_j v$ are bounded continuous functions as $M^{\infty,1} \subset C \cap L^\infty$, and by induction it follows that $v^{(\alpha)} \in C \cap L^\infty$ for every α. This proves (1).

(2) Since $u_{h_0} \in M^{1,\infty}$ by Theorem 4.1, and $S_0^0 \subset M^{\infty,1}$, it follows that $b = a * u_{h_0} \in M^{\infty,1}$. We shall prove that $b \in S_0^0$. By Taylor's formula it follows that if e is a unit vector in $\mathbf{R}^{2m}$, then $(b(X+\varepsilon e) - b(X))/\varepsilon$ is equal to

$$((\partial_e a) * u_{h_0})(X) + \varepsilon R_\varepsilon(X),$$

where

$$R_\varepsilon(X) = \int_0^1 (1-t)((\partial_{ee}^2 a)(\cdot + t\varepsilon e) * u_{h_0})(X)\,dt. \tag{4.9}$$

Since $D^\alpha a \in M^{\infty,1}$ for every α and that $u_{h_0} \in M^{1,\infty}$, it follows that each term in (4.9) belong to $M^{\infty,1}$, and that $\|R_\varepsilon\|_{M^{\infty,1}} \leqslant C$, for some constant C, independent of e and ε. Hence $|\varepsilon|\|R_\varepsilon\|_{L^\infty} \to 0$ as $\varepsilon \to 0$, since $M^{\infty,1} \hookrightarrow C \cap L^\infty$. This proves that $b \in C^1$ and that $\partial_e b = (\partial_e a) * u_{h_0} \in M^{\infty,1}$. It follows now by induction that $\partial^\alpha b = (\partial^\alpha a) * u_{h_0} \in C \cap L^\infty$ for every α. Hence $b \in S_0^0$, and the proof is complete. $\square$

We also have the following result.

Proposition 4.13. *Assume that* $t \in \mathbf{R}$ *and that* $p \in [1, \infty]$. *Then the following are true*:

(1) *if* $a \in M^{\infty,1}(\mathbf{R}^{2m})$, *then* $a_t(x, D)$ *is continuous from* $L^p(\mathbf{R}^m)$ *to* $B^{p,\max(p,p')}_{-m|1-2/p|}(\mathbf{R}^m)$;

(2) *if* $a \in S_0^0(\mathbf{R}^{2m})$ *and* $\mu > 1$, *then* $a_t(x, D)$ *is continuous from* $L^p(\mathbf{R}^m)$ *to* $H^p_{-\mu m|1-2/p|}(\mathbf{R}^m)$.

Proof. (1) Let $q = \max(p, p')$. From Proposition 1.7 and Theorem 3.1 it follows that $L^p(\mathbf{R}^m) \subset M^{p,q}(\mathbf{R}^m) \subset B^{p,q}_{-m|1-2/p|}(\mathbf{R}^m)$, and the result is a consequence of Theorem 4.3, if we let $p_0 = \infty$, $q_0 = 1$, $p_1 = p_2 = p$ and $q_1 = q_2 = q$.

Assertion (2) now follows from (1), (3.2) and that $S_0^0 \subset M^{\infty,1}$. $\square$

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